Gravitating discs around black holes

V Karas,[1,2] **J-M Huré**[3,4] **and O Semerák**[2]

[1] Astronomical Institute, Academy of Sciences of the Czech Republic, Boční II, CZ-141 31 Prague, Czech Republic
[2] Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, CZ-180 00 Prague, Czech Republic
[3] Observatoire de Paris-Meudon, Place Jules Janssen, F-92195 Meudon, France
[4] Université Paris 7 (Denis Diderot), 2 Place Jussieu, F-75251 Paris, France

Abstract. Fluid discs and tori around black holes are discussed within different approaches and with the emphasis on the role of disc gravity. First reviewed are the prospects of investigating the gravitational field of a black hole–disc system by analytical solutions of stationary, axially symmetric Einstein's equations. Then, more detailed considerations are focused to middle and outer parts of extended disc-like configurations where relativistic effects are small and the Newtonian description is adequate.

Within general relativity, only a static case has been analysed in detail. Results are often very inspiring, however, simplifying assumptions must be imposed: ad hoc profiles of the disc density are commonly assumed and the effects of frame-dragging and completely lacking. Astrophysical discs (e.g. accretion discs in active galactic nuclei) typically extend far beyond the relativistic domain and are fairly diluted. However, self-gravity is still essential for their structure and evolution, as well as for their radiation emission and the impact on the environment around. For example, a nuclear star cluster in a galactic centre may bear various imprints of mutual star–disc interactions, which can be recognised in observational properties, such as the relation between the central mass and stellar velocity dispersion.

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1. Introduction

The interplay between gravitational attraction and non-gravitational forces produces systems with various shapes in the universe, including elongated structures, spheroids, discs, shells and rings. All of them participate in variety of processes and act on different scales, ranging from cosmological dimensions down to planetary sizes. Some configurations, spheroidal in particular, hold by themselves, while other are maintained by an imposed control of a central body or an external field. This paper deals with discs and tori — typical configurations of rapidly circulating fluids that are governed by a central object. Its gravitational attraction supports global stability of the configuration, at least below a certain critical density of the disc, while compactness of the centre suggests prospective observational methods. High density configurations

suffer from instabilities and their mass contributes to general-relativity effects jointly with the central mass.

Nowadays it should be self-evident that black-hole discs are of indisputable relevance to different research communities, ranging from pure relativists to astrophysicists and observational astronomers. This diversification is reflected in varied approaches that people adopt to the problem. A mathematically minded theorist would take more interest in non-trivial (but still manageable) analytical solutions to Einstein's equations endowed with axial symmetry. An observer will want to actually *see* gas flows swirling in an accretion disc near a black hole (BH) or a compact star. Astronomers often invoke accretion discs to explain spectral features observed in galactic nuclei and in close (interacting) binary stars, although detailed physics of accretion flows still remains uncertain. Therefore, it is tempting to search for common intersections in such a broad subject. However, we will argue that serious distinctions cannot be neglected which prevent one to develop an all-encompassing scheme that would describe inherently different configurations. In particular, effects of general relativity (GR) have a chance to govern only the innermost regions of accretions discs and their role is quickly diminished with distance from the centre. Local (non-gravitational) physics plays crucial role and it often dominates over gravitation and governs observable properties of real systems (perhaps the most profound example arises if one attempts to compare accretion discs around stars with those around super-massive BHs). In order to see where limitations of any unified scheme stem from, it is useful to discuss both simplified and general analytical models and some particular astronomical objects in more detail.

In this paper, two-dimensional models are discussed. Needles to say, this is already enormous restriction. Obtained solutions offer a convenient tool to elucidate physical mechanisms connected with non-spherical sources of strong gravitational field. Compact toroids represent a non-trivial model of such objects that require GR, although Newtonian approach provides a sufficiently accurate description in many cases. Interaction with the central field is especially intriguing if the latter is generated by an ultra-compact body.

The paper starts with a summary of astrophysical motivations (section 2). Our attention is focused on discs around super-massive black holes, because in this case the importance of self-gravity has been traditionally acknowledged, but we also mention compact binary stars, in which case it has been recognised relatively recently that self-gravity may also play a role, especially in the context of neutron star collisions, tidal disruption and subsequent emission of high-energy gamma-rays and, possibly, gravitational waves. In section 3, the problem of massive discs around black holes is discussed within GR. Whereas GR is mainly relevant for the inner parts of self-gravitating discs (within one hundred Schwarzschild radii, $\lesssim 10^2 r_{\mathrm{s}}$), the field of actual discs is equally important for their structure at larger distances from the centre where the Newtonian regime is adequate (section 4). Evidence for self-gravity in accretion discs is still rather tentative and incomplete; some possibilities and observational implications are explored in section 5. There we also indicate the potential role of other ingredients, such as a compact nuclear star cluster in which the accretion flow may be embedded. However, the reader should be aware that important parts of the whole debate remain untouched in the present paper. In particular, we do not discuss magneto-hydrodynamical processes, neither do we address non-axially symmetric modes of the disc flow. One should bear in mind that these processes are probably crucial for understanding the mechanism of viscosity as well as launching of

jets in astronomical objects.

Disc structures are ubiquitous in such objects where the fluid circulates around and inflows onto a central compact body. The central mass, $M_\bullet$, can be considered as one of the model parameters and varied by many orders.[1] Its value provides primary classification of cosmic objects. Physical characteristics of accretion discs scale roughly with $M_\bullet$. Indeed, accretion discs around super-massive black holes in active galactic nuclei and quasars share *some* properties with circumstellar discs in close binary systems (e.g., cataclysmic variable stars, X-ray binaries and micro-quasars). For example, both the discs around stellar-mass black holes and accretion discs around super-massive black holes may develop local and global instabilities which involve gravity. However, there are important distinctions between these two kinds of objects which prohibit any simplistic scaling. It turns out that galactic nuclear discs tend to be cooler and less dense compared to circumstellar discs, which translates to different opacity of the material, its ionization structure and, hence, different spectral properties. There has been also a very lively and still open discussion, whether the presence of a black hole in stellar-mass BH candidates (or, rather, the absence of any hard surface) can be revealed in the radiation properties of their discs.

Environmental effects bring more distinctions between different objects with accretion discs. Unfortunately, at this point a bias is always inevitably introduced to our models by assumptions about turbulent transport of matter and momentum. Most solutions are model dependent. Neglecting this dependence can hardly be avoided when showing results of computations, nevertheless, we attempt to accentuate generic features when possible.

We refer to monographs by Frank, King & Raine [113] and Kato, Fukue & Mineshige [170] for exposition of accretion theory, to Krolik [188] and Peterson [273] for discussion of accretion flows with the emphasis on active galactic nuclei, and to various review papers [7, 73, 174, 209, 266] summarizing recent advances as well as the problems encountered by traditional scenarios.

2. Astrophysical motivations and some early works

Accretion plays a crucial role in the processes of energy liberation and mass accumulation that take place in the core of astronomical objects. Disc-type accretion represents an important mode which is realized under suitable circumstances, defined by the global geometrical arrangements, local microphysics of the fluid medium and the initial conditions as well. Rotation is crucial and it puts discs, toroids and rings in the same category of centrifugally supported systems. By accretion discs one understands topologically toroidal fluid configurations orbiting around a central body and gradually drifting towards it, provided a mechanism to redistribute the angular momentum. The centre governs or at least substantially contributes to the gravitational field in which the medium evolves. The formation of the central body is presumed as being completed, although the accretion process gradually transfers mass and angular momentum onto the central body and thus contributes to its subsequent evolution.

It is worth noting that the genesis of our own solar system with a star in the centre was imagined, by Pierre-Simon Laplace and Immanuel Kant, as emerging out

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[1] We introduce $M_\bullet$ for the black-hole mass in order to maintain clear distinction from the disc mass. Both contributions add to the total mass of the system. Similar situation arises with angular momentum, as discussed later in the text.

of a turbulent, slowly spinning nebula with flattened, disc-like shape.[2] This was in the final years of the 18th century. The potential associated with a gravitating ring was examined in detail by F W Dyson [94]. Later, considerations associated with star and planet formation were the main driving force behind the initial development of accretion disc theory. Modern evidence for such configurations was reported as early as in 1930s by Baxandall [30] in the context of binary stars. In the attempt to interpret spectrograms of the famous β Lyrae binary system and in order to explain the origin of secondary lines, visibly displaced in their wavelength, "circulating currents in the outer envelope" were tentatively invoked together with a gaseous envelope that surrounds the larger component. Since then, interacting binary stars have been the prototypical systems for studying accretion discs.

The treatment of self-gravitating discs was pursued in Ostriker's Newtonian equilibria of uniformly rotating, polytropic, slender rings [258]. Later, self-gravitating configurations with realistic equation of state and opacity were constructed and the basic formalism for self-gravitating BH discs was given in GR [27, 46, 109]. In stellar binary systems, the gas in the disc is provided by a donor star via Roche-lobe overflow or stellar wind [180]. Accretion process helps to feed the central body. If conditions are suitable, the gas radiates and provides a way to observe and study the system. Historically, limiting cases were distinguished for methodological reasons and for the sake of simplification. When speaking of hydrodynamical non-self-gravitating accretion, the main assumptions concern stationarity, spatial symmetries that are imposed on the flow and on the gravitational field (spherical flows with negligible angular momentum, axially symmetric accretion onto a moving body, and disc-like planar accretion), and the sound speed profile (subsonic versus supersonic flows). See, e.g. [42, 112, 146, 147] and references cited therein. It turns out that behaviour of truly three-dimensional unsteady flows is rather different and more complex [91, 157].

Nearly spherical accretion occurs if the angular momentum content of accreted material is negligible (this case appears appropriate mainly for accretion onto an isolated body). In another limit, one ignores radial transport of the material and considers an axially symmetric toroidal configuration that preserves perfect rotation about the symmetry axis. Both these cases were treated in many works, assuming the test-fluid approximation (flow lines are determined by the central gravity and pressure gradients only). Toroidal topology of the fluid system introduces the main difference when compared with the physics of rotating stars (for the latter, see a text-book exposition of the subject by Tassoul [335]). On large scales, the accretion mechanism was proposed as the energy source in bright galactic nuclei in late 1960s (Lynden-Bell [214]). Since then the idea has been continuously developed and brought to a text-book level.

Self-gravity has global consequences on the disc shape, namely, it influences the location of its inner and outer edges, as well as the disc geometrical thickness. The effect can be roughly parameterised by the ratio of the disc mass to the central BH mass, $q \equiv M_{\rm d}/M_\bullet$ [3, 185]. Wilson [363] speculated about the importance of self-gravity (especially in cataclysmic binary systems), which would have direct observational consequences, but this possibility was later rejected on the basis of

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[2] "... We see a region of space extending from the centre of the Sun to unknown distances, contained between two planes not far from each other ..." (I Kant, *Allgemeine Naturgeschichte und Theorie des Himmels*, 1755). Independently, P S Laplace advanced and published a theory of Saturn's rings and the nebular hypothesis of the planetary system (*Mémoire sur la théorie de l'anneau de Saturne*, Mém. Acad. Sci., 1789; *Exposition du systeme du monde*, 1796).

more detailed arguments; it turns out that non-standard (massive) discs are ruled out in cataclysmic binaries [148]. Self-gravity is normally unimportant in binary star systems also because their discs have limited space to occupy. Under typical conditions (i.e. among well-known cosmic systems), the disc mass plays a role in the process of gradual fragmentation of proto-planetary nebulae, formation of planetary bodies, and their subsequent migration. (Recent observations, in particular with the Hubble Space Telescope, have demonstrated that discs are a by-product of star formation and also present during the birth of planets.) Also planetary rings are subject of self-gravitational instabilities — but we cannot tackle all these lively topics here (see [130, 193, 294, 359]). There has been a recent interest in high-density (neutron) tori, in which case the disc own gravity must be important. This kind of massive discs could be formed in the course of binary star coalescence or as a product of collapse to a BH [135, 243]. The question of stability and time evolution of these heavy, transient tori is essential. The idea was advanced in the framework of scenarios of failed supernovae and strongly magnetised tori [50, 218, 262, 350, 351], and an interesting possibility of intermittent accretion was hypothesised in the context of massive neutron tori that may be formed from a tidally disrupted compact star [371].

As mentioned above, gravitational field of heavy discs influences the global structure of the system, which in turn determines possible figures of equilibrium. Coming back to accretion discs in galactic nuclei, vertical structure of these discs is affected by their self-gravity especially in the middle and the outer regions (beyond a few hundreds $r_{\rm S}$, say) where the vertical component of the disc gravitational field exceeds the corresponding z component of the central field [316–318, 329]. The main aspect of discs' non-negligible gravitational effect on their own structure concerns the stability with respect to perturbations. Self-gravity was first examined in the context of accretion discs around super-massive black holes by Paczyński [260], who proposed that local (Jeans-type) self-gravitational instabilities are invoked if the density exceeds a threshold value. Turbulent fragmentation may contribute to viscosity and to heating of the disc medium. Even the formation of stars may be induced in this way [186, 319]. Collin & Zahn [76, 77] explored the possibility and the impact of star-formation in galactic-centre discs. Recently, also Sirko & Goodman [322] discussed consequences which the formation of massive stars in the disc may have on its heating and spectrum. Non-axisymmetric instabilities of accretion discs and tori have been widely investigated (see e.g. [39, 129, 267] and references cited therein). Their importance for modelling realistic accretion discs has been soon recognised, but we do not examine these types of instability here. If one considers strictly axisymmetric and stationary rotational motion, one finds idealized configurations, which provide useful insight into the problem of accretion discs; for example, the question of the role of mean stationary solutions. However, there is no doubt that these solutions must be rather different from real accretion flows; investigations of the latter has culminated in recent three-dimensional numerical simulations [91]. It is noteworthy here that we are allowed to treat both accretions discs in binary stars (with an accreting stellar-mass black hole) and in galactic nuclei, because common physical mechanisms operate in both cases. However, there are important distinctions which become particularly essential when realistic configurations are discussed. We will point out limitations of simple scaling between the two limiting cases later in the text, but great part of this paper concerns very simplified situations in which rough scaling according to the system size and mass is possible.

In the next section we summarize techniques that have been employed to derive
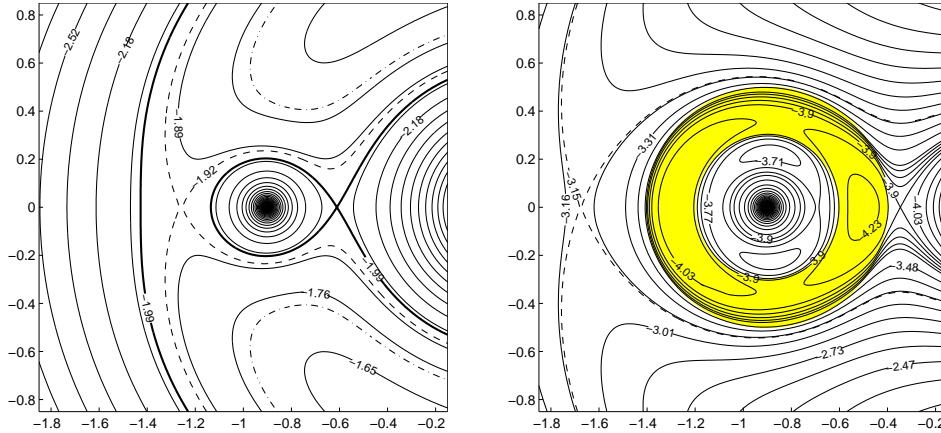
Figure 1. Left: Contours of the Roche potential are plotted for two point masses ("stars") in their orbital plane. Here, the ratio of masses is assumed to be 0.1 (coordinates are scaled by separation between the two components; no disc is present). Right: The same type of graph as on the left side, but with a contribution of a thin planar disc taken into account as an additional source of the gravitational field. The plot shows the potential structure in the critical region near to the cusp, where the mass exchange takes place. See [363] for an original discussion of the binary Roche potential with a massive disc taken into account.

spacetimes of gravitating discs around black holes within GR framework and then we proceed to possible astrophysical applications of massive discs. We start the discussion by briefly recalling a uncomplicated case of a light non-self-gravitating Newtonian disc that can be formed at a certain evolutionary stage of a binary star when the donor component dumps material on the other one. A text-book analysis of matter transfer employs the notion of Roche potential in which the actual gravitational field of a binary is approximated by the field of two massive points orbiting around a common centre of mass [45] (see the left panel of figure 1). An accretion disc forms when the donor fills its Roche lobe and starts overflow near the Lagrange L_1 point. The gas transferred in this way possesses high angular momentum and creates a disc-like structure before being captured by the accreting body. The Roche model gives a basic classification of binary stars into three main categories according to the sizes of individual components: detached, semi-detached and contact systems. As mentioned above, their discs can usually be treated as non-gravitating.

In contrast to this transparent scheme, a sufficiently massive disc could change the potential field near the cusp. In the right panel of figure 1, distortion of equipotential surfaces is depicted with the disc contribution to the gravitational field taken into account (here, the disc mass is comparable to that of the secondary star). It is relevant to remark that equilibria of relativistic tori also show the structure of critical lobes, which is similar to the Roche geometry, even if the fluid is non-self-gravitating and the only source of gravity is the central black hole (see section 3.3). This structure facilitates accretion onto the central body.

Equilibrium figures of critical (lobe-filling) tori represent a starting point for the discussion of accreting black holes in compact binary systems. Naturally, the Roche lobe geometry is applicable for the mass overflow onto a black hole as much as it is in other systems of close binaries, in which the mass is exchanged between the

components. However, topologically similar configurations of critical tori can arise also when accretion proceeds onto an isolated black hole. In this case the torus can develop a cusp and the overflow for a purely relativistic reason. The latter situation is particularly important for accretion onto super-massive BHs in galactic nuclei.

An issue of global stability of such configurations must be raised: if a small amount of material is transferred through the inner cusp onto the central body, then a new, modified distribution of mass and of angular momentum is established. Geometrical shape of the critical equilibrium is changed accordingly. A question arises whether the new configuration is stable or whether, instead, it undergoes further accretion in a catastrophically increasing (runaway) rate. Abramowicz *et al* [2] claimed the presence of this kind of instability in pseudo-Newtonian massive tori, while Wilson [364] demonstrated that mass-less tori (i.e. test fluid) around a Kerr BH are not run-away unstable. Answer to the problem of this instability depends on the radial distribution of angular momentum throughout the initial configuration (before the mass exchange takes place) and, in the relativistic case, also on BH angular momentum parameter. It was demonstrated that increasing BH rotation and a positive slope of the angular momentum distribution both have a stabilizing effect [6, 86, 110, 213]. Increasing M_d, on the other hand, acts against stability [228, 229, 253, 254]. Comparisons [111] are not easy because different approaches have been employed in papers (numerical techniques vs. analytical calculations; proper GR treatments vs. approximations and pseudo-Newtonian models). The black-hole–disc systems are difficult to treat exactly, even in the test-fluid approximation. Severe simplifications have thus been imposed on theoretical models, ranging from presumed symmetries and boundary conditions to various assumptions about microphysics of the medium. Simplifying assumptions are the more unavoidable when the disc gravity (playing a role of the external source to the gravitational field) can *not* be neglected and when one intends to proceed analytically. Further restrictions are dictated by applicability of the results to real cosmic objects.

3. Stationary axisymmetric fields in general relativity

There are at least three aspects in which the system of a rotating black hole with an external source attracts relativists. First, due to the non-linearity of Einstein's equations, the field of a multi-body system is a traditional challenge where, in most cases, one does not manage with a simple superposition. On the first post-Newtonian level, the "celestial mechanics" of gravitationally interacting bodies can be kept linear (see Damour *et al* [87, 88]), but in the strong-field regime the interaction may bring surprising features that have only been described in very few cases yet (for a two-body problem, the current "state of art" is the third post-Newtonian approximation [89]).

The second point concerns manifestation of frame dragging due to the rotation of the sources. Contrary to the Newtonian treatment (that does not discriminate directly between static and stationary situations), the relativistic field is determined not only by mass-energy configuration, but also by its motion within the bodies: the inertial space can loosely be imagined as a viscous fluid mixed by the sources. In today's "gravito-electromagnetic" language: gravity has not only an electric component, generated by the mass, but also a magnetic one, generated by mass currents [340]. Let us only refer to somewhat casual selection [68, 97, 163, 176, 216, 227, 271] for further details at this point.

Needless to say, the third point is the presence of a BH itself. Near such an extreme body, the deviations from Newtonian theory become dominant; in particular,

the above mentioned implications of non-linearity and rotation reveal themselves prominently.

The class of stationary axisymmetric solutions of Einstein equations is the appropriate framework for the attempts to include the gravitational effect of an "external" source in an exact analytical manner. At the same time, such spacetimes are of obvious astrophysical importance, as they describe the exterior of a body in equilibrium [158] (see [27, 51, 54, 59, 65, 97, 206, 360] for thorough expositions on stationary and axially symmetric spacetimes in general relativity).

In the Weyl–Lewis–Papapetrou coordinates (t, R, ϕ, z) of the cylindrical type, the stationary axisymmetric metric can be written as[3]

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + R^2 B^2 e^{-2\nu}(\mathrm{d}\phi - \omega\mathrm{d}t)^2 + e^{2\lambda-2\nu}(\mathrm{d}R^2 + \mathrm{d}z^2), \qquad (1)$$

where the unknown functions ν, B, ω and λ only depend on R and z; ω is interpreted as the angular velocity of inertial-frame dragging with respect to observers at rest at spatial infinity. The metric coefficients g_{ab} have invariant meaning, they can be expressed in terms of Killing fields $k^\mu = \partial x^\mu/\partial t$ and $m^\mu = \partial x^\mu/\partial\phi$:

$$g_{tt} = -e^{2\nu} + \omega^2 g_{\phi\phi} = k_\iota k^\iota, \qquad (2)$$

$$g_{t\phi} = -\omega g_{\phi\phi} = k_\iota m^\iota, \qquad (3)$$

$$g_{\phi\phi} = R^2 B^2 e^{-2\nu} = m_\iota m^\iota. \qquad (4)$$

The corresponding sub-determinant is $\det(g_{ab}) = -R^2 B^2$, so

$$g^{tt} = -e^{-2\nu}, \quad g^{t\phi} = -e^{-2\nu}\omega, \quad g^{\phi\phi} = -e^{-2\nu}\omega^2 + R^{-2}B^{-2}e^{2\nu}. \quad (5)$$

The Einstein equations read

$$\boldsymbol{\nabla} \cdot (R\boldsymbol{\nabla}B) = 8\pi RB\left(T_{RR} + T_{zz}\right), \qquad (6)$$

$$\boldsymbol{\nabla} \cdot \left(R^2 B^3 e^{-4\nu}\boldsymbol{\nabla}\omega\right) = -16\pi B e^{2\lambda-2\nu} T_\phi^t, \qquad (7)$$

$$\boldsymbol{\nabla} \cdot (B\boldsymbol{\nabla}\nu) = \tfrac{1}{2}R^2 B^3 e^{-4\nu}\left(\boldsymbol{\nabla}\omega\right)^2 + 4\pi B e^{2\lambda}\left(2T^{tt} + e^{-2\nu} T_\sigma^\sigma\right), \qquad (8)$$

where $\boldsymbol{\nabla}$ and $\boldsymbol{\nabla}\cdot$ stand for the gradient and divergence in a flat three-dimensional space with coordinates (R, ϕ, z); thus $\boldsymbol{\nabla}X = (X_{,R}, 0, X_{,z})$ and $\boldsymbol{\nabla} \cdot \boldsymbol{X} = R^{-1}[(RX^R)_{,R} + (RX^z)_{,z}]$ in the axially symmetric case. Once B, ω and ν are known, λ can be integrated from equations

$$
\begin{aligned}
B\lambda_{,z} - B_{,z} + R\left(B_{,R}\lambda_{,z} + B_{,z}\lambda_{,R} - B_{,Rz} - 2B\nu_{,R}\nu_{,z}\right) \\
+ \tfrac{1}{2}R^3 B^3 e^{-4\nu}\omega_{,R}\omega_{,z} = 8\pi RBT_{Rz},
\end{aligned} \qquad (9)
$$

$$
\begin{aligned}
2B\lambda_{,R} - 2B_{,R} + R\left[2B_{,R}\lambda_{,R} - 2B_{,z}\lambda_{,z} - B_{,RR} + B_{,zz} - 2B\left(\nu_{,R}^2 - \nu_{,z}^2\right)\right] \\
+ \tfrac{1}{2}R^3 B^3 e^{-4\nu}\left(\omega_{,R}^2 - \omega_{,z}^2\right) = 8\pi RB\left(T_{RR} - T_{zz}\right).
\end{aligned} \qquad (10)
$$

The unknown metric functions are subject to boundary conditions on the horizon (if there is one), on the symmetry axis and at spatial infinity. The metric must be regular on the horizon and on the axis, and in the case of an isolated source one also requires asymptotic flatness (the conditions are discussed in [27, 54], for example).

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[3] Geometrized units are used in which $c = G = 1$; the signature of the metric tensor $g_{\mu\nu}$ is $(-+++)$. Greek indices run from 0 to 3 and Latin indices $(i, j, \dots)$ run from 1 to 3; indices from the beginning of the Latin alphabet $(a, b, \dots)$ will represent cyclic coordinates t and ϕ. Partial differentiation is denoted by ∂ or by a subscript comma, covariant derivative is denoted by ∇.

Outside of the sources, where $T_{\mu\nu} = 0$, the simplest solution of equation (6) with a satisfactory asymptotic behaviour (equation (26) below) is $B = 1$. The two remaining field equations (7)–(8) reduce to

$$\boldsymbol{\nabla} \cdot (R^2 e^{-4\nu} \boldsymbol{\nabla} \omega) = 0, \tag{11}$$

$$\boldsymbol{\nabla}^2 \nu = \tfrac{1}{2} R^2 e^{-4\nu} (\boldsymbol{\nabla}\omega)^2 \tag{12}$$

and the relations (9)–(10) yield

$$\lambda_{,R} = R \left(\nu_{,R}^2 - \nu_{,z}^2 \right) - \tfrac{1}{4} R^3 e^{-4\nu} (\omega_{,R}^2 - \omega_{,z}^2), \tag{13}$$

$$\lambda_{,z} = 2R\,\nu_{,R}\,\nu_{,z} - \tfrac{1}{2} R^3 e^{-4\nu}\,\omega_{,R}\,\omega_{,z}. \tag{14}$$

Equations (11)–(12) are often presented in the form of the Ernst equation [98]

$$- g_{tt} \boldsymbol{\nabla}^2 \mathscr{E} = (\boldsymbol{\nabla}\mathscr{E})^2 \tag{15}$$

for a complex (Ernst) potential $\mathscr{E} \equiv -g_{tt} + \mathrm{i}\psi$ whose imaginary part ψ is given by

$$R\,\psi_{,z} = g_{tt}\,g_{t\phi,R} - g_{tt,R}\,g_{t\phi}, \qquad R\,\psi_{,R} = g_{tt,z}\,g_{t\phi} - g_{tt}\,g_{t\phi,z}. \tag{16}$$

λ is found by a line integration as above.

3.1. Essential features of the stationary axisymmetric spacetimes

A black-hole horizon is a null hypersurface below which the spacetime is dynamical (see e.g. [54, 55, 117] for thorough accounts). In Weyl-Lewis-Papapetrou coordinates, the horizon is located where $\det(g_{ab}) = -R^2 B^2$ vanishes. In the vacuum case we can put $B = 1$, so the horizon lies on the axis then, $R = 0$. The BH interior thus has to be studied in different coordinates, e.g. in spheroidal coordinates of the Boyer–Lindquist type (t, r, θ, ϕ), introduced by the transformation

$$R = \sqrt{\Delta}\,\sin\theta, \qquad z = (r - M_\bullet)\,y, \tag{17}$$

where $\Delta = (r - M_\bullet)^2 - k^2$ and $y = \cos\theta$, $M_\bullet$ being a scale parameter (it represents the black-hole mass) and $(\pm)k$ determining where the horizon reaches up along the z axis. The horizon is now given by $\Delta = 0$, i.e. $r = M_\bullet + k \equiv r_{\mathrm{H}}$ (hence, $z = ky$; there exist more black-hole horizons in general, but we mean the outermost one here).

Important parameters of the horizon are its surface area A, surface gravity κ_{H} and the angular velocity ω_{H} relative to infinity,

$$A = 2\pi \int\limits_0^\pi \sqrt{(g_{\theta\theta}\,g_{\phi\phi})_{\mathrm{H}}}\,\mathrm{d}\theta, \quad \kappa_{\mathrm{H}} = |\nabla e^\nu|_{\mathrm{H}}, \quad \omega_{\mathrm{H}} = \omega(r = r_{\mathrm{H}}); \tag{18}$$

κ_{H} and ω_{H} are constant all over the horizon.

In order to learn a "true shape" of the BH, not distorted by coordinates, the horizon must be represented properly as a two-dimensional surface in a three-dimensional Euclidean space. One can e.g. follow [323]. The two-dimensional metric is first rewritten as

$$\mathrm{d}s^2 = \frac{A}{4\pi} \left[h^{-1}(y)\,\mathrm{d}y^2 + h(y)\,\mathrm{d}\phi^2 \right], \tag{19}$$

where $h(y) = 4\pi A^{-1}(g_{\phi\phi})_{\mathrm{H}}$. An isometric embedding of the two-surface (y, ϕ) in $\mathbb{E}^3$ with coordinates (X, Y, Z) is then given by

$$X = \frac{A}{4\pi}\sqrt{h}\,\cos\phi, \qquad Y = \frac{A}{4\pi}\sqrt{h}\,\sin\phi, \qquad Z = \frac{A}{4\pi}\int\limits_0^y \sqrt{\frac{1}{h}\left(1 - \frac{1}{4}h_{,y}^2\right)}\,\mathrm{d}y. \tag{20}$$

If the horizon's Gaussian curvature

$$\mathscr{C}_{\mathrm{H}} = -8\pi^2 A^{-2} h_{,yy} \tag{21}$$

is negative in some place, the global embedding is impossible.

On the symmetry axis ($R = 0$), the regularity condition requires $e^\lambda = B$, and therefore the metric adopts the form $\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + B^2 e^{-2\nu}\mathrm{d}z^2$ here. Simplification of the formulae can also be expected in the equatorial plane ($z = 0$) if the spacetime is reflectionally symmetric with respect to it. This is usually the case for astrophysically motivated considerations.

In the static case $\omega = 0$. If $T^R_R + T^z_z = 0$ (which is fulfilled with zero pressure), the metric (1) acquires the Weyl canonical form

$$\mathrm{d}s^2 = -e^{2\nu}\mathrm{d}t^2 + R^2 e^{-2\nu}\mathrm{d}\phi^2 + e^{2\lambda - 2\nu}(\mathrm{d}R^2 + \mathrm{d}z^2). \tag{22}$$

Equations (6)–(10) reduce themselves to Poisson's equation

$$\boldsymbol{\nabla}^2 \nu \equiv R^{-1}\nu_{,R} + \nu_{,RR} + \nu_{,zz} = 4\pi e^{2\lambda - 2\nu}(T^\phi_\phi - T^t_t) \tag{23}$$

for ν, and to relations

$$\begin{aligned}
&\lambda_{,z} - 2R\nu_{,R}\nu_{,z} = 8\pi R T_{Rz}, \\
&\lambda_{,R} - R\left(\nu_{,R}^2 - \nu_{,z}^2\right) = 4\pi R\left(T_{RR} - T_{zz}\right)
\end{aligned} \tag{24}$$

for λ. The linearity of Laplace equation allows for simple superposition of vacuum Weyl fields.

From the astrophysical point of view, a given analytical solution is problematic if it contains features which should not be present in a realistic source, such as (i) physical singularities on or above the horizon, or (ii) bad asymptotic behaviour. The existence of a singularity follows — according to its type [69, 95] — e.g. from the divergence of scalars constructed from the metric tensor or from its derivatives, or from the divergence of physical (tetrad) components of the Riemann tensor. The Kretschmann invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is usually checked first. However, the spacetimes are known which do contain singularities, although all of their curvature invariants vanish. It is thus difficult to verify the *non*-existence of singularities; the only proof of regularity at a given point is to find a coordinate system in which the metric is smooth enough locally.

Astrophysically relevant is the region of outer communications outside the horizon, but a relativist is interested in the black hole interior as well, because even there it is possible to perform physical observations. In the Weyl–Lewis–Papapetrou coordinates the metric is not defined below the horizon (this would correspond to imaginary radius R) and must be extended there. One asks about how is the BH interior (and the singularity in particular) influenced by the presence of external sources. The answer is unknown even for quite simple superpositions.

Far away from the horizon, one inquires for the asymptotic behaviour of the field. That of an isolated stationary source falls to zero in a specific manner — the spacetime is said to be asymptotically flat (e.g. chapter 11 in R Wald's book [358], or section 3 in J Ehler's Festschrift [32]). In case of the metric (1), it must hold, for $r \to \infty$,

$$\nu = -\frac{\text{total mass}}{r} + \mathscr{O}(r^{-2}), \tag{25}$$

$$B = 1 + \mathscr{O}(r^{-2}), \tag{26}$$

$$\omega = \frac{2 \cdot (\text{total angular momentum})}{r^3} + \mathscr{O}(r^{-4}), \tag{27}$$

$$\lambda = \mathcal{O}(r^{-2}). \tag{28}$$

Although a suitable combination of solenoidal motions can satisfy the assumption of stationarity and axial symmetry, it is natural to assume that the elements of the source follow the simplest type of motion — that along spatially circular orbits ($R =$ const, $z =$ const) with steady azimuthal angular velocity $\Omega = \mathrm{d}\phi/\mathrm{d}t$. Such a motion follows the symmetries, so an observer on a circular orbit experiences time-independent field around. In order that the corresponding four-velocity $u^\mu = u^t(k^\mu + \Omega m^\mu)$ points inside the light cone, $(u^t)^{-2} = -g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi} = e^{2\nu}(1 - \hat{v}^2)$ must be positive (we denoted $\hat{v} = RBe^{-2\nu}(\Omega - \omega)$ the linear speed with respect to the local zero-angular-momentum observer). The latter condition is fulfilled if Ω falls within the range

$$\Omega_{\min} \equiv \omega - \frac{e^\nu}{RB} \leq \Omega \leq \omega + \frac{e^\nu}{RB} \equiv \Omega_{\max}. \tag{29}$$

On the horizon (where $e^\nu = 0$), the permitted range narrows down to a unique value ω_{H}. In the above-given equations, the four-velocity covariant components

$$\begin{aligned}
\gamma &\equiv -u_t = -u^t(g_{tt} + \Omega g_{t\phi}) = u^t e^{2\nu} + \omega\ell, \\
\ell &\equiv u_\phi = u^t g_{\phi\phi}(\Omega - \omega)
\end{aligned} \tag{30}$$

stand for the specific energy and the specific azimuthal angular momentum with respect to an observer at spatial infinity.

In case of the circular orbit, the four-acceleration $a_\mu = u_{\mu;\nu}u^\nu$ can be written as

$$a_\mu = -\tfrac{1}{2}g_{\alpha\beta,\mu}\,u^\alpha\,u^\beta = -\frac{g_{tt,\mu} + 2\Omega\,g_{t\phi,\mu} + \Omega^2 g_{\phi\phi,\mu}}{2e^{2\nu}\,(1 - \hat{v}^2)}. \tag{31}$$

It has at most two non-zero components, a_R and a_z; in general, a_z is zero in the equatorial plane and non-zero elsewhere, while a_R vanishes if

$$\Omega = \Omega_\pm = \frac{-g_{t\phi,R} \pm \sqrt{(g_{t\phi,R})^2 - g_{tt,R}\,g_{\phi\phi,R}}}{g_{\phi\phi,R}}, \tag{32}$$

where the upper/lower sign corresponds to a co-rotating/counter-rotating orbit (these terms may be somewhat misleading in spacetimes with large angular momentum). Among equatorial geodesics, three particular cases are important: the photon, the marginally bound and the marginally stable circular orbits. The photon orbits delimit the regions where the circular motion is time-like; they are given by equalities $\Omega_\pm = \Omega_{\max}^{\min}$. The marginally bound orbits demarcate the regions where particles on circular orbits have lower energy than necessary for the existence at spatial infinity; the limiting case is given by $\gamma(\Omega = \Omega_\pm) = 1$. The marginally stable orbits also bound the regions where circular motion is stable with respect to small perturbations acting within the orbital plane. Rayleigh's criterion of linear stability gives the location of the marginally stable orbits where $[\ell(\Omega = \Omega_\pm)]_{,R} = 0$. These orbits play a key role in the theory of accretion discs: the matter is swept away from unstable sectors. In particular, the innermost marginally stable orbit should represent the disc inner rim. However, perturbations *perpendicular* to the orbital plane can also be important.[4] For the above-mentioned exact solutions of Einstein's equations, the frequencies of vertical oscillations were given explicitly [311].

[4] It has been discussed quite recently that (non-self-gravitating) accretion discs are prone to warping due to extraneous irradiation [281]. This effect is, naturally, modified when self-gravitation is taken into account [269].

A peculiar feature of rotating fields are the dragging effects. In accordance with Mach's ideas of relativity of motion and inertia (and his interpretation of Newton's bucket experiment in particular), Lense & Thirring showed, in the early times of general relativity, that dragging is present in Einstein's theory. Machian inspiration remains alive within contemporary relativity — see, for example, the last-decade references [68,163,176,215,216] or the Tübingen conference [25] and references therein. The effect has obvious analogy in electromagnetism, where (electric) currents generate the magnetic component of the field. An extreme implication of dragging is the occurrence of the ergosphere in the vicinity of ultra-compact rotating objects. In this region $\Omega_{\min} > 0$, thus the stationary observer cannot remain static (i.e. at rest relative to infinity), albeit he still withstands the radial attraction. The static-limit surface, given by $\Omega_{\min} = 0$ and hence by $g_{tt} = 0$, also represents the set of points from where signals emitted by static observers ($\Omega = 0$) reach infinity with an infinite redshift. Several processes working in the ergosphere have been suggested, by means of which the rotational energy of the black hole could be extracted without violating the second law of BH dynamics [283].

With the metric (1), $g_{tt} = 0$ corresponds to $RB\omega = e^{2\nu}$ which can only be satisfied by $R \geq 0$, so the static limit really lies outside the horizon;[5] in particular, $R = 0$ implies $e^{2\nu} = 0$, so the static limit touches the horizon at the axis.

3.2. Sources

In order to complete the relativistic solution, one must also describe the interior of sources — where $T_{\mu\nu} \neq 0$. It is difficult to find a realistic interior solution, especially if it has to match smoothly the vacuum exterior and if we restrict ourselves to analytical approaches. The problem can be substantially simplified by assuming that the source is infinitely thin: the solution is then vacuum-type everywhere, with the energy-momentum tensor $T_{\mu\nu} = g_{zz}^{-1/2} S_{\mu\nu} \delta(z)$ found from the discontinuity of the normal field across the source as in electrodynamics (the appropriate covariant method is known as the Israel's formalism [29, 159]).

The case of a thin equatorial layer (a disc) in a stationary axisymmetric spacetime has notably been studied by [126, 197]. The disc lying on $z = $ const has zero radial pressure inside ($S_{RR} = 0$). At the same time, the vertical pressure vanishes ($S_{zz} = 0$) if the disc mid-plane coincides with the equatorial plane ($z = $ const $= 0$) of the reflectionally symmetric spacetime. In the Weyl–Lewis–Papapetrou coordinates, the non-zero $S_{\mu\nu}$ components read [197]

$$S_{ab} = -\frac{\sqrt{g_{RR}}}{8\pi} \left(\frac{g_{ab}}{g_{RR}} \right)_{,z}. \tag{33}$$

Using (5) and (14), this yields

$$S_t^t = -\frac{e^{\nu-\lambda}}{8\pi} \left[4\nu_{,z}(1 - R\nu_{,R}) - R^2 e^{-4\nu} \omega_{,z}(\omega - R\omega_{,R}) \right], \tag{34}$$

$$S_\phi^t = -\frac{e^{\nu-\lambda}}{8\pi} R^2 e^{-4\nu} \omega_{,z}, \tag{35}$$

$$S_t^\phi = -\frac{e^{\nu-\lambda}}{8\pi} \left[4\omega\nu_{,z} - (1 + R^2 e^{-4\nu} \omega^2)\omega_{,z} \right], \tag{36}$$

[5] In coordinates which also describe the BH interior, *two* horizons and *two* static limits are found in general. We mean the outer horizon and the outer static limit everywhere.

$$S^\phi_\phi = \frac{1}{8\pi} R e^{\nu-\lambda} \left[4\nu_{,R}\nu_{,z} - Re^{-4\nu}\omega_{,z}(\omega + R\omega_{,R}) \right]. \tag{37}$$

The total mass and angular momentum of the disc are fixed by the Komar integrals that, respectively, lead to

$$M_{\rm d} = \frac{1}{2} \int_b^\infty g^{tc} g_{tc,z} R \,\mathrm{d}R = -\frac{1}{4} \int_b^\infty \frac{g^2_{\phi\phi}}{R} \left(\frac{g_{tt}}{g_{\phi\phi}} \right)_{,z} \mathrm{d}R, \tag{38}$$

$$J_{\rm d} = \frac{1}{4} \int_b^\infty g^{tc} g_{c\phi,z} R \,\mathrm{d}R = \frac{1}{4} \int_b^\infty \frac{g^2_{\phi\phi}}{R} \omega_{,z} \,\mathrm{d}R; \tag{39}$$

$R = b > 0$ is the position of the disc inner rim. All the z-derivatives are understood to be calculated in the limit $z \to 0^+$.

If a BH is present in the disc's centre, its contribution to the total mass and angular momentum need to be included. For a given type of spacetime, the aggregate values of the parameters can be written as sums, $M_\bullet + M_{\rm d}$ and $J_\bullet + J_{\rm d}$ (cf. [55], chapter 6.6.1). The black-hole parts can be calculated as Komar integrals, or they can be inferred by subtracting $M_{\rm d}$, $J_{\rm d}$ from the totals, derived from the metric asymptotes (25,27). $M_\bullet$ and $J_\bullet$ are related through Smarr's formula,[6]

$$M_\bullet = 2\omega_{\rm H} J_\bullet + \frac{1}{4\pi} \kappa_{\rm H} A = 2\omega_{\rm H} J_\bullet + k \,. \tag{40}$$

Further, if the term

$$D \equiv \left(S^\phi_\phi - S^t_t \right)^2 + 4 S^t_\phi S^\phi_t = \frac{e^{2\nu-2\lambda}}{16\pi^2} \left(4\nu_{,z}^2 - R^2 e^{-4\nu}\omega_{,z}^2 \right) \tag{41}$$

is non-negative, then a stationary observer exists with respect to whom the energy-momentum tensor assumes a diagonal ("isotropic") form

$$S^{\mu\nu} = \hat{w} u^\mu_{\rm iso} u^\nu_{\rm iso} + \hat{P} v^\mu_{\rm iso} v^\nu_{\rm iso}, \tag{42}$$

where $u^\mu_{\rm iso} = u^t_{\rm iso}(1,0,\Omega_{\rm iso},0)$ and $v^\mu_{\rm iso} = -R^{-1}(\ell_{\rm iso},0,\gamma_{\rm iso},0)$ are, respectively, observer's four-velocity and the unit basis vector in the ϕ-direction. Angular velocity of this observer is

$$\Omega_{\rm iso} = \frac{1}{2S^t_\phi} \left(S^\phi_\phi - S^t_t - \sqrt{D} \right) = \omega - \frac{2e^{4\nu}\nu_{,z}}{R^2\omega_{,z}} + \sqrt{\left(\frac{2e^{4\nu}\nu_{,z}}{R^2\omega_{,z}} \right)^2 - \frac{e^{4\nu}}{R^2}} \,; \tag{43}$$

$\gamma_{\rm iso}$ and $\ell_{\rm iso}$ denote the corresponding specific energy and angular momentum at infinity.[7] The observer measures the surface density

$$\hat{w} = \frac{\gamma^2_{\rm iso} S^{tt} - \ell^2_{\rm iso} S^{\phi\phi}}{u^t_{\rm iso}(\gamma_{\rm iso} + \Omega_{\rm iso}\ell_{\rm iso})} \tag{44}$$

and tangential pressure

$$\hat{P} = \frac{R^2 u^t_{\rm iso}(S^{\phi\phi} - \Omega^2_{\rm iso} S^{tt})}{\gamma_{\rm iso} + \Omega_{\rm iso}\ell_{\rm iso}}. \tag{45}$$

[6] See Carter [55], equation (6.297), for derivation and discussion. For the original formulation in Kerr spacetime, see [324].

[7] The case of $D < 0$ represents discs with non-zero heat flow. Then the energy-momentum tensor must be written in a more general form, $S^{\mu\nu} = \hat{w} u^\mu_{\rm iso} u^\nu_{\rm iso} + \hat{P} v^\mu_{\rm iso} v^\nu_{\rm iso} + \hat{K}(u^\mu_{\rm iso} v^\nu_{\rm iso} + v^\mu_{\rm iso} u^\nu_{\rm iso})$, with $\hat{K} = \sqrt{-D/2}$ and $\Omega_{\rm iso} = (2S^t_\phi)^{-1}(S^\phi_\phi - S^t_t) = \omega - 2R^{-2}\omega^{-1}_{,z}e^{4\nu}\nu_{,z}$. See [126].

If $\hat{w} \geq \hat{P} > 0$, the energy-momentum tensor can represent two equal streams of particles moving on (accelerated) circular orbits in opposite directions at the same speed $\sqrt{\hat{P}/\hat{w}}$. This "speed of sound" was shown [126] to be equal to the geometric mean of the local prograde and retrograde circular geodesic speeds $\hat{v}_\pm$ as measured by the observer u_{iso}^μ:

$$|\hat{v}_+\hat{v}_-| = \hat{P}/\hat{w}, \tag{46}$$

where

$$\hat{v}_\pm = \frac{(v_{\text{iso}})_t + (v_{\text{iso}})_\phi \Omega_\pm}{(u_{\text{iso}})_t + (u_{\text{iso}})_\phi \Omega_\pm} \; ; \tag{47}$$

$\Omega_\pm$ are the angular velocities of the prograde and retrograde circular geodesics (32). Although counter-rotating relativistic streams are hardly applicable to any realistic astrophysical situation, they can be very illuminating especially in the discussion of physical sources of the Kerr metric and other spacetimes.[8]

Further physical requirements can be raised, namely, the weak energy condition ($\hat{w} \geq 0$, $\hat{P} \geq -\hat{w}$), the dominant energy condition ($\hat{w} \geq 0$, $|\hat{P}| \leq \hat{w}$) and the non-negativity of pressure ($\hat{P} \geq 0$). Their combination gives $\hat{w} \geq \hat{P} \geq 0$. The requirements are very simple in the static case: if $\nu_{,z}(z = 0^+) > 0$, then $\hat{P} \geq 0$ is equivalent to $\nu_{,R} \geq 0$ which is satisfied below the Lagrangian point of zero field (where $\nu_{,R} = 0$ and $\hat{v}_\pm^2 = 0$). In regions with tension ($\hat{P} < 0$) the particles would be attracted more by the outer parts of the disc than by the central BH, thus no circular geodesics exist (then hoop stresses have to be employed to interpret the disc matter); this typically happens in discs with much matter on large radii. The other condition, $\hat{w} \geq \hat{P}$, is equivalent to $\hat{v}_\pm^2 \leq 1$. To summarize, the above constraints are in fact ensured by the obvious condition $0 \leq \hat{v}_\pm^2 \leq 1$ which can be written explicitly as $1 \geq 1 - R\nu_{,R} \geq 1/2$.

3.3. Solving Einstein's equations for a black hole with matter around

Relativistic spacetimes are being found in three ways: (i) by numerical solution of Einstein's equations, (ii) by perturbation of previously known spacetimes, and (iii) by exact analytical solution of Einstein's equations. Let us mention several results of the above approaches on stationary axisymmetric spacetimes that describe rotating black holes with additional matter.

Teukolsky [338] lists today's computational resources and the appropriate, hyperbolic formulation of the field equations as the main "reasons why we are on the verge of important advances in the computer solution of Einstein's equations". Whereas analytical prospects are restricted, the present-day computers can handle almost any situation. However, covering a sufficiently representative time-span is still a problem. Furthermore, it is often impossible to recognise whether a given numerical solution represents a typical situation or just a marginal case. Due to this

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[8] Bičák & Ledvinka [35] have constructed pressure-less counter-rotating discs in the Kerr geometry, employing Israel's technique [29, 159]. Analogous approach was used by other authors in a different context of exact solutions; see e.g. Abramowicz, Arkuszewski & Muchotrzeb [1], Turakulov [347], and Pichon & Lynden-Bell [275], or a particularly lucid paper by Bičák, Lynden-Bell & Katz [37]. These works may appear rather academic for people who are interested mainly in astrophysical applications of gravitating discs. However, a very pertinent question is addressed in these papers, namely, under what conditions a material disc could represent a source for the Kerr metric. Very recently, the "displace, cut and reflect" method has been extended to generate static thin discs with halos [355] and static *thick* discs [127].

lack of generality, it is difficult to discern, analyse and interpret different classes of solutions across boundless ranges of possibilities. Nevertheless, numerical solutions provide explicit *examples* of spacetimes and processes that might otherwise remain only conjectural [198].

Numerical spacetimes with a rotating BH surrounded by a stationary axisymmetric source were already constructed by Lanza [191] (a hole with a thin finite equatorial disc) and by Nishida & Eriguchi [252] (a hole with a thick toroid); recently, a new ("multidomain spectral") method has been utilized to spheroidal as well as toroidal fluid configurations [16], with the possibility of inclusion of a central object in future. To mention just one particular point, contrary to a common experience (also gained by [252]) that the horizons inflate towards the external sources, [191] ended with a prolate horizon (stretched along the rotation axis) in certain cases (when the disc was strongly counter-rotating with respect to the hole). It would be an interesting consequence of the interplay between dragging from the hole and from the disc if it were confirmed that this can indeed happen, even though under extreme circumstances only.

A great deal of literature and several formalisms have been devoted to perturbations of BH spacetimes. Will [360] gave an explicit result for the perturbation of the Schwarzschild metric by a (rotating) axisymmetric weakly gravitating thin equatorial ring. This direct approach is however not practicable for a rotating hole and/or for an extended source with pressure (not mentioning more complicated situations).

The rotating case (specifically, the algebraically special vacuum case) was discussed notably by Teukolsky [337] who succeeded in separating the decoupled equations for the first order perturbations of a Kerr BH into the second-order ordinary differential equations. By solving the "gravitational" Teukolsky equation, the perturbative deformation of the Kerr horizon was calculated by [90]. The stationary axisymmetric Green's function of the Teukolsky equation was provided by [210]. The most comprehensive expositions of the first-order BH perturbation theory were given by Chandrasekhar [60, 61] in the Newman–Penrose formalism.

In a related approach to the first-order perturbations of rotating fields [66, 171, 343, 345], the perturbation components are expressed in terms of a single (Debye) potential that obeys a wave-like equation (adjoint of the Teukolsky master equation). The problem of perturbation of the Kerr horizon was treated this way by Chrzanowski [67]. Yet another approach, not restricted to algebraically special backgrounds, was advanced in [301]. Note that all these perturbative methods deal with curvature components, as opposed to the earlier metric formalisms of Regge & Wheeler, Zerilli and Moncrief.

In the charged case, the perturbation problem introduces coupling between gravitational and electromagnetic quantities. Conversion between gravitational and electromagnetic waves takes place. In the case of Reissner–Nordström BH, these interacting perturbations were studied by Bičák [33] and, more recently, by Torres del Castillo *et al* [344]. A gauge invariant derivation of the basic equations was given by Fernandes & Lun in Schwarzschild space-time [106], while [107] discussed a rotating analogy and provided its link with the Teukolsky equation.

The second-order perturbation theory has also been under development since the 1970s. For Schwarzschild black holes, it is surveyed in Gleiser *et al* [122], while the rotating case is tackled in Campanelli & Lousto [52].

It should be remarked that solutions describing stationary sources around black

holes are not the primary aim of the perturbation strategy. Historically, attention has rather been devoted to the stability of black-hole solutions, to the relaxation of black holes into a stationary no-hair state, to non-stationary processes of astrophysical significance (such as the behaviour of weakly gravitating particles or waves in BH backgrounds) and to gravitational waves (see Thorne [339] for a survey).

One can add that the approximate (though non-perturbative) method for superposing two (rotating) black holes was proposed by Krivan & Price [187] to fix initial data for both analytic and numerical computations of BH collisions.

A perturbative approach is adequate in situations where the outside source has only a very small effect on the field of the main body. Whenever this is not the case, the superpositions have to be described by solutions of the full, non-linear theory. At the end of Bičák's survey [34], one is encouraged "not to cease in embarking upon journeys for finding them, and perhaps even more importantly, for revealing new roles of solutions already known", because "Is there another so explicit way of how to learn more about the rich possibilities embodied in Einstein's field equations?" More recent reviews of vacuum fields can be found in Bonnor *et al* [43, 44]. For further details, see also the canonical monograph on exact solutions to Einstein's field equations by Stephani *et al* [328].

Until now, no explicit exact solutions describing the system of a rotating centre with an additional axisymmetric ring, disc or torus have been found. Nevertheless, many stationary axisymmetric solutions of the (electro-)vacuum Einstein equations are known that do generalise those containing only isolated black holes. These were mostly obtained by indirect methods known as "generating techniques". Two major approaches, developed by the end of the 1970's — the group-theoretic techniques and the soliton-theoretic (or inverse-scattering) techniques, work for spacetimes with two commuting symmetries. (See Kordas [181] for a review and [80–82, 144, 219] for more detailed analysis and interrelations between different formulations, e.g. those of Kinnersley & Chitre; Maison; Belinskii & Zakharov; Harrison; Hoenselaers, Kinnersley & Xanthopoulos; Hauser & Ernst; Neugebauer; Kramer & Neugebauer; or Alekseev.) Other related methods have been proposed more recently, e.g. the simplification of the Hauser–Ernst integral equation by Sibgatullin [320], "monodromy data transform" by Alekseev [9] (also [179]), static gravitational multipoles [131] (cf. [284]) and their superposition with stationary fields [132] by Gutsunaev & Manko, linear transformation by Quevedo [284] or "finite-gap" (algebraic-geometric) solutions by Korotkin & Matveev [182]. Most of them have been compared in reference [10]. (For other approaches, see e.g. [71,334] or the results of Nakamura and Kyriakopoulos, referred to and worked out by [336]; cf. also [352].)

The crucial point of the soliton generating techniques is a solution of two linear differential equations (Lax pair) the integrability conditions of which are exactly the Einstein equations (namely the Ernst equation). The linear problem can be tackled in order to generate new solutions from the known ones: given some ("seed") metric with two Killing vectors, it yields a different metric of the given type (the procedure is often called Bäcklund transformation in analogy with the technique used for the KdV equation). In such a manner, many known spacetimes were reproduced, but also broad families of new solutions were provided characterised by arbitrarily large sets of free parameters. Only a very restricted number of them have been given a clear physical interpretation, however. Though several of these results perhaps represent a rotating BH in an "external" gravitational field (e.g. [47, 62, 63, 220, 221, 285]), none of the latter has yet been firmly linked with a concrete realistic body such as ring, disc

or torus (but cf. section IV in Letelier & Oliveira [204]). Namely, when performing the Bäcklund transformation, it is difficult to require specific physical properties of the spacetime being generated — one rather tries what will come out. See [284] for a hint of how to overcome this. More recently, Letelier [203] interpreted a solution obtained from a Weyl seed by the inverse-scattering transformation with an even number N of solitons as a superposition of this Weyl solution — i.e. of a given set of static multipoles — with $N/2$ Kerr(-NUT) solutions, represented by $N/2$ rotating axial bars in Weyl-type coordinates.

As an example, let us give the attempt [372] which was "cultivated" from a general Weyl metric by a real-two-soliton version of the Belinskii-Zakharov method. The resulting metric was written as a generalisation of the Kerr(–NUT) solution and many of its properties were proven satisfactory. Choosing the inverted first Morgan–Morgan disc (see below) as a seed, various characteristics of the superposition were plotted [307] against the rotation parameter of the central hole, the mass of the disc and its inner radius. The characteristics simplify very much in the limit when the BH becomes extreme. One of the interesting features of this limit is the vanishing of the "external" gravitational field on the horizon [305]. This effect of expulsion of the external (stationary axisymmetric) fields from rotating (and/or charged) black holes, analogous to the Meissner effect in (super)conductors, was observed in magnetic fields before (see the recent survey [36]). However, a serious problem has been found in the equatorial plane: the black-hole horizon has a sharp narrowing there. This is caused by the appearance of a supporting surface between the hole and the source outside it [307]. Indeed, the possible problems with physical interpretation of the solution (e.g. negative density or pressure or matter moving at a superluminal speed) also involve the occurrence of supporting singular surfaces or lines ("struts") that indicate that a given system of bodies cannot remain in stationary (or static) equilibrium.

The struts may equally well "spoil" a static superposition, though this case almost reduces just to adding the two potentials, $\nu = \nu_1 + \nu_2$. The singularity may appear in calculating the second metric function,

$$\lambda = \int\limits_{\text{axis}}^{R,z} R\left[\left(\nu_{,R}^2 - \nu_{,z}^2\right)\,\mathrm{d}R + 2\nu_{,R}\,\nu_{,z}\,\mathrm{d}z\right]. \tag{48}$$

Consequently, even the *static* axisymmetric case has only been afforded a few realistic superpositions. Let us refer to two of them involving a Schwarzschild BH: the superposition with an infinite annular thin disc, obtained by Lemos & Letelier [201] by inversion of the first counter-rotating finite disc of Morgan & Morgan [241] and the one with an inverted isochrone thin disc, obtained by Klein [177]. The Morgan–Morgan family of static axisymmetric finite thin counter-rotating discs [241] is given by Newtonian surface density

$$w^{(m)}(R \leq b) = \frac{(2m+1)M_{\mathrm{d}}}{2\pi b^2}\left(1 - \frac{R^2}{b^2}\right)^{m-1/2} \qquad (m = 1, 2, \ldots). \tag{49}$$

Since the zeroth member is clearly singular at the rim, most attention was devoted to the first ($m = 1$) member. It appeared to be a prototype of a simple and physically meaningful Weyl field and it was also considered as a limit case of a more general class of *stationary* discs [14, 115]. The corresponding annular case, obtained by the inversion $R \to \frac{b^2 R}{R^2 + z^2}$, $z \to \frac{b^2 z}{\rho^2 + z^2}$, was shown to really allow for physically satisfactory situations [201, 310, 312, 313, 370]. There is nevertheless a problem at the rim of

these discs, already suspected by checking the radial derivatives of (49): the m-th member of the family has infinite m-th and higher derivatives of the density at the outer rim. Calculation of the Kretschmann curvature invariant reveals that the rim is really singular and that the singularity is inherited by the inverted discs [304, 370]. The superpositions of a Schwarzschild BH with (inverted) "higher" Morgan–Morgan counter-rotating discs were studied in [308]. Comparison of the results obtained for the first ten of them showed that the properties of the disc are very sensitive to the local density profile. It will be desirable to also consider different classes of annular thin discs in superpositions, preferably of those not having any such drawback at the rim.

We have already explored one different family of annular disc solutions, namely, the case when density is a power-law in Weyl radius,

$$w^{(m,n)}(R \geq b) = \frac{\prod\limits_{l=1}^{m}(ln+1)}{m! \; n^m} \frac{M_{\mathrm{d}}b}{2\pi R^3} \left(1 - \frac{b^n}{R^n}\right)^m . \tag{50}$$

However, these discs also turned out not to be regular at the rim; $(m+1)$-st radial derivative of their potential diverges, although all the radial derivatives of density are finite there, and the first $m-1$ derivatives even vanish. In a different context of galactic stellar discs, an interesting approach has been developed by Sibgatullin, Garcia & Manko [321] who examined a new method of reconstruction of the surface density distribution of a finite size disc with a central BH.

Allowing for self-gravitation of the matter around the horizon, one has to reduce the generality of the problem elsewhere. Whereas test discs are treated within stationary axisymmetric settings, analytical solutions with exact inclusion of the ambient material have only been studied in more detail in static cases yet. This is of course a serious limitation, because the collapsed cores of galaxies as well as ultra-compact components in binaries are likely to rotate, perhaps very rapidly, which means that inertial dragging should be important there in addition to the effects of pure scalar attraction. Moreover, the composite spacetimes presented in the literature are constructed with discs whose field can be expressed in a tractable form rather than with those which really follow from some model of accretion. However, some superpositions — such as e.g. those involving thin annular discs obtained by inversion of the Morgan–Morgan solutions [201, 310–313, 370] — were shown to possess physically acceptable properties within an astrophysically plausible range of parameters. As such, they can indicate some of the effects that could occur as a consequence of non-negligible gravitational contribution of the disc (see [306] for a more thorough survey).

Let us return to the linear problem (Lax pair of linear equations) associated with the Ernst equation. In order to control the physical properties of the solution being constructed, it is desirable to translate the physical boundary conditions of the Ernst equation into the language of quantities which appear in the linear problem and then solve the latter (rather than to apply a "random" Bäcklund transformation to some metric [202]). This leads to Riemann–Hilbert problems known from the theory of completely integrable differential equations [17, 179, 249]. Tackling the stationary axisymmetric boundary value problem with the help of the linear system, the field of two physically relevant types of sources has been discussed: that of black holes and that of finite thin discs ([14, 115, 232, 250, 251] and references therein). The hope that the above methods could be used to describe superpositions of both (e.g. [250, 251]) has recently been supported by Klein [178] who presented an exact space-time of a

regular black hole surrounded by an infinite annular disc as a subclass of Korotkin's solutions to the Ernst equation, obtained by Riemann-surface techniques [183]. The resulting metric is given in terms of theta-functions, being asymptotically flat under suitable choice of parameters. Klein's solution seems to involve cases with physically acceptable source disc.

The limit case should be briefly mentioned when gravitation of the external matter is ignored and the central rotating BH fully determines the field. Then the spacetime geometry is described by Kerr metric and the equilibrium configurations can be found analytically [26,109], at least if the fluid obeys a simple equation of state (such as a barytropic relation) and remains in dynamical equilibrium with purely rotational motion (four-velocity $\boldsymbol{u} = u^\phi \boldsymbol{e_\phi} + u^t \boldsymbol{e_t}$). This case is over-simplified from the astrophysical point of view, but it captures several crucial relativistic effects which do not occur in the Newtonian limit — namely, the possibility of fluid overflow across the inner edge, which acts, effectively, as the Lagrange L_1 point in a binary system.

The relativistic Euler equation can be integrated in terms of specific enthalpy, $W \equiv -\int (\epsilon + P)^{-1} dP = \ln |u_t| - F(l)$, which can be further expressed as a function of $l(R, z)$, the profile of angular momentum density [4, 5, 12, 185]. The only unknown function that needs to be specified is the rotation law, either in terms of $l = -u_\phi / u_t$ (which is a conserved quantity), or angular velocity $\Omega = u^\phi / u^t$. Rotation can be differential but the choice is not completely arbitrary because, according to Rayleigh's criterion for linearly stable configurations, l must be increasing monotonically outwards from the rotation axis (a borderline case of constant l was discussed in detail). W plays the role of an effective potential for inertial forces and determines equilibrium configurations in which the equipotential and isobaric surfaces coincide, so this way constant pressure surfaces can be found. For example, in a special case of marginally stable configuration with $l = $ const, one finds $W(R, z) = \ln |u_t|$.

The fluid can establish a stationary configuration provided that the W-isosurface is closed (see figure 2), otherwise overflow and subsequent accretion take place. Let us recall that equilibrium configurations with their elongated cross-section (bounded by the critical, self-crossing lobe) *resemble* the shape of Roche lobes. The possibility of fluid transport across the cusp occurs when the critical lobe is filled; accretion can thus proceed without a necessary action of viscosity. However, the two situations are of very different nature. In the former case, it is the combined gravitational effect of the binary together with (Newtonian) centrifugal force which define the lobes, while, in the latter case, the only source of gravity is the central body with relativistic effects playing a crucial role.

3.4. Gravitational effects of black-hole discs

Compared with the vacuum case (with a BH but without a disc), modifications of the observable characteristics can be linked with the presence of self-gravitating matter. Various approaches have been adopted in order to construct solutions and discuss motion in the gravitational field of black-hole discs. One can start, e.g., by constructing equilibrium structures of self-gravitating polytropic thick discs [40]. Formalism was developed also for perturbative solution of a (slowly rotating, weakly gravitating, geometrically thin) ring around a Schwarzschild BH [360,361]. Furthermore, equatorial motion was explored in pseudo-Newtonian studies [172] of a rotating BH surrounded by a massive thin equatorial ring, and in GR analysis of the corresponding static
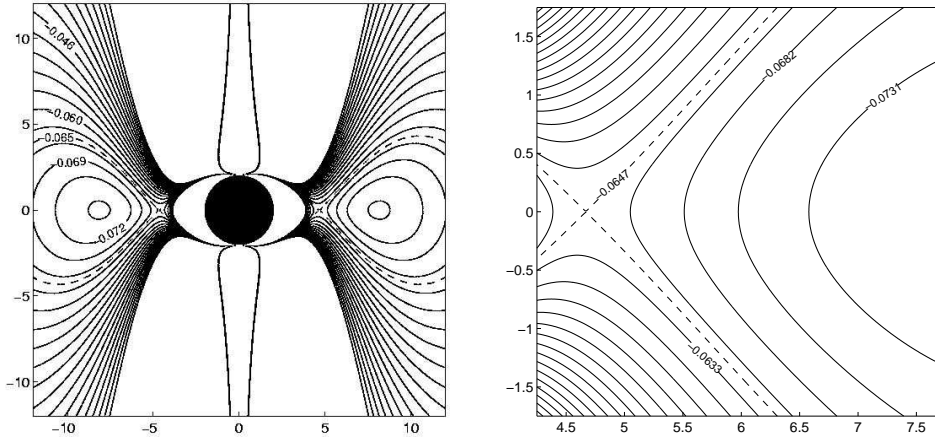
Figure 2. The equipotential surfaces of effective potential $W(x,z)$ are shown in a meridional section (x,z) of the Schwarzschild metric. Left: Contours of $W = \mathrm{const}$ are plotted with solid lines. The critical self-crossing contour, $W \doteq -0.065$, is indicated with a dashed line. Equilibrium is possible if the fluid (in pure rotational motion) fills one of the closed surfaces. Notice that these configurations can extend far from equatorial plane. The black hole horizon is indicated by a circle of radius $r_\mathrm{S} = 2M$ around the origin. Right: A detail is shown of the structure of contours near the cusp. Fluid overflow provides the possibility of relativistic accretion across the cusp without the need for viscosity. This effect also stands behind the origin of the runaway instability [5, 185].

case [57].[9] The accretion-disc self-induced field was also discussed in connection with the disc's vertical structure and stability. It turns out that certain modes of instability can be damped whereas others are amplified by the disc's gravity [13, 268]. Fully relativistic results employ numerical spacetimes [191, 252]. Within these solutions, the computation of emission-line profiles were performed; the case of moderately massive thin finite discs was treated by [164], and that of heavy toroids by [348].[10] Another issue is the oscillation of the disc material excited by a small perturbation in the horizontal (R and ϕ) or vertical (z) direction [169]. In the static case, the formulae

[9] The pseudo-Newtonian approach was devised in order to introduce effects of general relativity into accretion models [263]. The black hole is treated as a Newtonian body whose gravitational field is determined by Paczyński–Wiita potential, $-M_\bullet/(r - r_\mathrm{S})$. This simple expression mimics the gravitational field of a non-rotating BH; in particular, it reproduces the radii of marginally stable and marginally bound circular orbits of free test particles. Different variants of the pseudo-Newtonian potential were proposed in order to include rotation of the BH and to obtain better agreement with relativity (see [309] for references). The pseudo-Newtonian potential does not aspire to represent any gravitational theory, and the corresponding potential is not required to satisfy any field equations. This fact, however, does not diminish the practical value of the model, which applies even to regions with strong gravity and captures qualitative features of motion near the horizon.

[10] Let us mention here two examples of the effects that we observed recently in studying the properties of the Lemos–Letelier superposition. By generating a sequence of superpositions parameterised with relative disc mass, and demanding that (i) all the disc matter be interpretable as two equal counter-rotating streams of particles on stable time-like equatorial circular geodesics and that (ii) the inner disc rim be fixed at the least possible radius, it was found [310] that heavier discs of the family can reach considerably closer to the horizon not only in Schwarzschild r, but also in terms of physical measures — circumferential radius $r \exp(-\nu_\mathrm{d})$ and proper radial distance from the horizon.

for the frequencies of free horizontal and vertical oscillations give [311][11]

$$\kappa^2 = \frac{e^{4\nu - 2\lambda}}{1 - R\nu_{,R}} \left(\nu_{,RR} + 4R\,\nu_{,R}^3 - 6\nu_{,R}^2 + 3R^{-1}\nu_{,R} \right), \tag{51}$$

$$\Omega_\perp^2 = \frac{e^{4\nu - 2\lambda}}{1 - R\nu_{,R}} \left[\nu_{,zz} - 4\nu_{,z}^2 \left(1 - 2R\nu_{,R} \right) \right]. \tag{52}$$

Without the external source, $\Omega_\perp^2$ reduces to the square of the Schwarzschild orbital frequency, $\Omega_\pm^2 = M(M + \sqrt{R^2 + M^2})^{-3}$, whereas κ^2 remains different due to the pericentre precession. It was found that perturbations in the horizontal direction are more important for the stability of light discs, whereas the vertical perturbations are more dangerous for heavy discs: for superpositions of the given type, the curve of marginal vertical stability is more restrictive than that of marginal horizontal stability for $M_\mathrm{d} > 0.2296M$. It even fully excludes discs with $M_\mathrm{d} \geq 2M/7$ [370]. Disc self-gravity makes oscillations of the inner parts faster and the region of the horizontal mode trapping somewhat smaller.

When the black hole and the disc are allowed to rotate, manifestations of self-gravity could be even more pronounced, because in relativity the kinetic energy (as well as any other form of energy) also generates the field. Corresponding mass-energy currents give rise to frame dragging. *Whereas the mass of the black hole is probably dominant in a real accretion system, the disc can bear much of the angular momentum, thus modifying the gravito-magnetic field of the central object significantly.* This could be decisive for the mechanisms of rotational energy extraction. Note, however, also the possibility of having accreting systems where the disc/torus mass can actually be comparable to that of the central black hole and which can represent a transient state of gravitational collapse. In this context, the most popular system is a close binary of ultra-compact objects. Late evolutionary stages of such systems (in-spiral, merger and the ring-down phase) and the fate of debris have been studied, numerically and perturbatively, as a tentative source of gravitational waves (see [149] and references therein) and also as an engine behind gamma-ray bursts [86, 246, 282, 296, 298]. In particular, at late stages of the neutron star–black hole binary in-spiral, the neutron star could tidally break up [229, 235, 254, 339, 350].

Two-dimensional toroidal shapes similar to the one shown in figure 2 were originally invented mainly as a model for large and very diluted flows which probably exist in quasars and similar objects. These considerations have been continuously refined and discussed in various context (recently, e.g. [156] and references cited therein). As mentioned above, already in early works it was recognised that self-gravity imposes limits on this scheme and should be taken into account in calculations. Self-gravity is also relevant for a recently proposed idea [371] that non-stationary tori could be formed in accreting binaries. A relatively small size and steep pressure gradients of these configurations put them in a corner of parameter space where they may show high-frequency quasi-periodic oscillations [349], which makes them very attractive to astronomers, but formation of such structures still needs to be addressed.

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[11] The perturbations are considered non-gravitating here, i.e. the spacetime metric is not perturbed. Allowing the perturbation to generate its own self-consistent field would only have a second-order effect on the frequencies.

4. Astrophysical discs in Newtonian regime

From the point of view of gravitational theory, and also for the nature of the accretion flow, the most important part is the inner region near to the BH. However, the gravitational field of actual discs has substantial and even, possibly, dominant influence on their structure also at large distances (exceeding several $\times 10^2 r_{\rm S}$; in physical units, $r_{\rm S} = 2GM_\bullet/c^2 \dot{=} 2.95 \times 10^5 (M_\bullet/M_\odot)\,{\rm cm}$) where the self-gravity is comparable to the central field. Newtonian physics is sufficient at those distances. The importance of the disc self-gravity is often checked by Toomre's parameter, $Q = \kappa c_{\rm s}/(\pi\Sigma)$: the local stability criterion with respect to axially symmetric perturbations requires $Q > 1$ [341, 342] ($c_{\rm s}$ is a local sound speed). Spiral structures may be produced already at $Q \sim 2$–4. Depending on the radial distribution of the thermodynamic quantities, global instabilities can indeed occur and play a key role in enhancing the accretion rate [209]. (Of course, at this point one must recall the theories of galactic dynamics [38].) This may facilitate angular momentum transport and provide a feeding mechanism for central black holes [118, 141, 318]. Again, we concentrate on discs in galactic nuclei although similar mechanisms are relevant also for discs around stellar-mass black holes and, to a certain degree, even for circumstellar discs in star-forming regions.

4.1. Effects of self-gravity derived from simple arguments

In addition to the classical equations of mass, momentum and energy conservation, the description of Newtonian gravitating fluids must include Poisson's equation,

$$\Delta\nu_{\rm d}(\boldsymbol{r}) = 4\pi\rho(\boldsymbol{r'}), \tag{53}$$

which links the mass density distribution ρ in the disc and the self-gravitating potential $\nu_{\rm d}$ ($\boldsymbol{r'}$ refers to the source and $\boldsymbol{r}$ is the field point). To understand the signatures of self-gravity expected in the fluid structure and motion, one can consider a purely rotating, steady-state system with zero viscosity and no significant radial pressure gradient. The radial component of the Euler equation (in cylindrical coordinates) is then: $\Omega^2 R = \nabla_R\,\nu_\bullet + \nabla_R\,\nu_{\rm d}$. In the absence of rotation, steady solutions are forbidden and the fluid falls towards the centre in a short time scale. Free fall can be avoided thanks to the centrifugal term which compensates the total gravitational attraction $g_R \equiv -\nabla_R(\nu_{\rm d} + \nu_\bullet)$. If radial self-gravity is unimportant, that is if $\nabla_R\nu_{\rm d} \ll \nabla_R\nu_\bullet$, then rotation of the fluid is determined by the BH mass only, $\Omega^2 \sim \Omega_{\rm K}^2 = \frac{1}{R}\nabla_R\nu_\bullet$. This assumption underlies most disc models in astrophysics. In particular, Newtonian geometrically thin disc models fall within this category [280]. In the opposite situation, it is evident from the simple form of Euler's equation that, as the disc carries some mass, angular velocity is expected to increase in order to prevent accretion. This implies that (radially) self-gravitating discs are expected to rotate faster than non-self-gravitating ones.

From the previous estimate one can conclude that heavy (fluid) discs should exhibit a super-Keplerian rotation curve (in terms of $\Omega(R) > \Omega_{\rm K}(R)$).[12] Keplerian rotation could still be maintained in a self-gravitating disc, but this would be a surprising result of some fine tuning among various quantities — pressure gradients, advection of momentum, effects of turbulent transport, etc. Neither can the possibility

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[12] It was demonstrated [153] that the rotation curve of maser spots in NGC 1068 could be explained by gravitational attraction of the disc around the central BH.

of sub-Keplerian rotation be ruled out. Such a case could occur in the outermost disc regions if $\nabla_R \nu_{\mathrm{d}} < 0$, for instance due to the presence of a dense region near the edge or due to significant negative pressure gradients. Further, viscous stresses can switch the accretion.

Let us now analyse the vertical component of the Euler equation in the same idealized situation as described above (again in the Newtonian limit): $\rho^{-1}\nabla_z P = -\nabla_z \nu_\bullet - \nabla_z \nu_{\mathrm{d}}$. This is hydrostatic equilibrium in z-direction which tells us that central gravity acts against pressure gradients and tends to gather material around the equatorial plane of the disc. For a bare central point mass, the magnitude of the vertical component of gravity decreases with radius. As an inevitable consequence, the disc tends to 'flare' (its thickness being roughly proportional to the distance from the axis, or even steeper, and it can wildly oscillate in the vertical direction). The atmosphere of the disc (i.e. its less dense region away from the equatorial plane) gets thicker where the gravity is weaker. The geometrical semi-thickness of the disc then satisfies the relation $\mathscr{R} \equiv \frac{\mathrm{d}\ln H}{\mathrm{d}\ln R} > 0$. Models show such flaring in the non-self-gravitating regime. Regarding the vertical direction, the disc self-gravity is significant if $\nabla_z \nu_{\mathrm{d}} \sim \nabla_z \nu_\bullet$. For any fluid element this results in enhancement of the vertical gravity; thus, a vertically self-gravitating disc should be geometrically thinner and more dense than a non-self-gravitating one. As will be shown below, some accretion disc models even yield $\mathscr{R} < 0$ in those regions, so in these circumstances the disc thickness cannot exceed a certain limiting value. This is important for the disc irradiation (and self-irradiation).

Some fundamental properties of discs and tori can be understood in the apparently simple framework of Maclaurin spheroid theory. Early analytical investigations by Poincaré, Dyson and, later, by Chandrasekhar [58] have been supplemented with recent numerical investigations (Hachisu [133,134]). These works shed light on possible shapes and dynamical characteristics of steady, self-gravitating polytropes in the inviscid limit. Both axially and non-axially symmetrical equilibrium patterns have been discussed. It turns out that equilibrium sequences can describe any conceivable system (depending on the polytropic index of the gas): these include multiple star systems, tri-axial (Jacobi) ellipsoids, some very strange shapes ('pear', 'amonit', etc.), but also discs, tori and rings. In the case of global solid-body rotation, discs, tori, and rings occupy the bottom-right part in the usual diagram of rotation frequency versus angular momentum. Furthermore, in a recent paper by Ansorg *et al* [15] a sequence of compressible rings is investigated with high precision. A bifurcation occurs from the Maclaurin sequence to the multi-ring sequence. What remains to be explored is the case of a full multi-ring sequence and its limit for the number of rings approaching infinity. An interesting connection could appear with the theory of disc fragmentation into individual rings [123]. The theory of Maclaurin spheroids and rings is built from simple arguments and is fundamental for understanding the problem of gravitating discs, but it still needs to be enriched with other physical ingredients.

Disc modelling poses a difficult task even in the Newtonian regime. In addition to the technical embarrassment of solving the Poisson equation with sufficient accuracy, difficulties are connected with strong coupling between magnetohydrodynamic terms and radiative transfer in the medium, and with our ignorance of boundary and initial conditions (they cannot be sufficiently constrained by observations but are decisive for the evolution of flows).[13] Furthermore, huge values of the Reynolds number

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[13] Note that even in the standard (non-self-gravitating) thin disc model, the black-body temperature of the disc medium does not scale proportionally to the central mass. The enormous diversity in

(typically $\gtrsim 10^{15}$) are associated with accretion flows. In consequence, simulations do not currently reach the numerical resolution required to investigate simultaneously both the large-scale streams and the smallest swirls where kinetic energy is dissipated. This is another reason why (magneto-)hydrodynamical simulations of accretion flows should be interpreted with caution [212]. Despite the tremendous complexity of these problems, models and simulations have reached a fair level of credibility [140]. Thermal equilibrium of the disc and its possible fragmentation have been also explored [120].

Given the technical obstacles inherent in the problem, disc modellers often resort to a semi-phenomenological parameterisation of the turbulent transport by using a viscosity coefficient of the form $\nu_t = \alpha c_s H$, where H is the disc half-thickness and $\alpha \leq 1$ is a parameter. The α-parameterisation, originally advocated in the context of binary accretion discs by Shakura & Sunyaev [314], remains a recipe enabling study of the characteristics of a mean accretion flow at the lowest level of approximation. However, it does not explain the process of triggering and maintaining the turbulence. Other viscosity laws have been proposed, for instance the β-prescription which is based on analogy with turbulence observed in laboratory sheared flows and gives $\nu_t \propto v_\phi R$ [217, 295]. Also, there is the prescription of marginally stable ($Q = $ const) discs with $\nu_t \propto v_\phi H^2/R$, designed for weakly (locally) self-gravitating discs [28]. Very probably turbulent transport and its effects cannot be satisfactorily mimicked through simple prescriptions. Below we touch on several properties which can be derived from the classical α-prescription, as well as the properties of $Q = $ const discs. Given the current knowledge, one cannot decide which of these approximations is better.

4.2. Poisson's equation in discs

Association of the density field $\rho(\boldsymbol{r})$ with a prescribed Newtonian potential is straightforward and always possible. Various classes of density–potential pairs can be formed [38, 100]. Unfortunately, the reverse procedure, i.e. the inversion of Poisson's equation, is difficult in general. No analytically tractable expression is available for ν_d that would correspond to a general three-dimensional system of finite size and mass. Exact solutions corresponding to non-spherical systems cannot be written generally in a closed form if the matter distribution is two-dimensional, non-homogeneous or of non-trivial shape or topology. Astrophysical discs exhibit all three complications.

As an illustration we show first, in figure 3 (left panel), the density distribution in the standard disc model where self-gravity effects are *not* included [152]. The complexity of shape and density distribution is striking: discs appear to be complex objects for which the merit of special analytical (and especially self-similar) solutions is very limited or doubtful.

An alternative approach is to consider the integral definition for the potential,

$$\nu_d(\boldsymbol{r}) = -\int_{\text{disc}} \frac{\rho(\boldsymbol{r}')}{|\boldsymbol{r}' - \boldsymbol{r}|} \, \mathrm{d}^3 r', \tag{54}$$

and to use expansions of Green's function (for instance in cylindrical harmonics or another kind of equivalent infinite series representation). This approach is often employed in order to find self-similar solutions and to investigate the disc stability [150, 196]. However, as pointed out in [70], convergence of the series is not guaranteed

opacity, spatial and time scales, etc. does not allow for bare scaling between corresponding physical quantities from one system to another. As a result, *Galactic* BH *candidates have rather different discs from those around super-massive* BH*s in galactic nuclei.* Distinctions and common properties of these two fiducial cases of accretion flows have been reviewed by Mirabel [237] and Czerny [85].
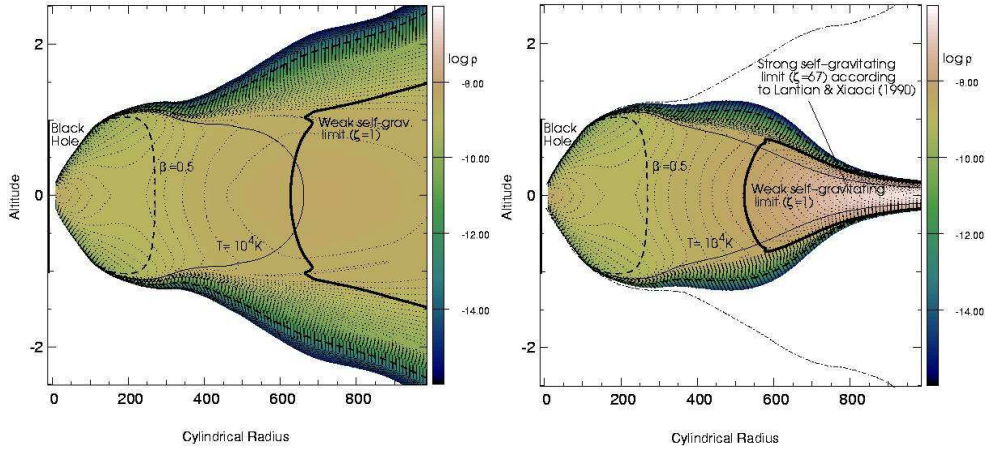
Figure 3. The shape and density distribution of an accretion disc feeding a $10^8 M_\odot$ black hole. Left: the WSG limit (cf. section 4.3) is indicated by $\zeta(R, z) = 1$ (thick curve). The limit is reached in the equatorial plane as close as $\sim 600 r_{\rm S}$. At this point and beyond, the disc self-gravity is expected to play a dominant role in the structure and dynamics of the flow (the SSG limit and the unstable limit are out of range adopted in this plot). Right: The case of the SSG limit is shown. A standard-type steady-state model was employed [152,280] (the viscosity parameter is $\alpha = 0.1$ and the accretion rate $\dot{M} = 0.1 M_\odot \, {\rm yr}^{-1}$). Lengths are plotted linearly and normalised to the Schwarzschild radius; the density scale is in g/cm^3.

in all points of matter distribution. Also, the expansion leads to uncertainties because the inferred series must be truncated.

In order to be able to treat realistic systems, one has to seek numerical solutions. However, it is difficult to achieve sufficient computational accuracy using standard techniques. Why is this? The first reason is that no boundary conditions are available in the vicinity of the disc; instead, they can be imposed only at infinity ($\nu_{\rm d} \to 0$ for $r \to \infty$). Secondly, the physical boundary of the fluid is not known *a priori* and it can evolve with time. One cannot rely on the standard situation that is met within laboratory physics of a fluid confined within walls, in which case the boundary conditions are imposed by experimental settings. The third difficulty is caused by the relatively large and variable aspect ratio: H/R may be as large as unity and even greater at some parts of the disc, and as low as $\sim 10^{-5}$ at other places (cf. figure 3). This fact hinders discussion of discs in active galactic nuclei, where the geometrical thickness of the disc can vary wildly over its diameter.

There are two practical ways of solving the problem numerically [242]. The first one employs straightforward determination of the potential from equation (54) everywhere in the disc. This approach can be very accurate if one is able to cope with integrable singularities inside the source ($\boldsymbol{r} \to \boldsymbol{r'}$) and it can account for very distorted disc shapes and large density gradients. However, the computations involved are generally very time consuming. One can envisage some cell-based technique [325] (as done in particle simulations [143]), but this is not expected to reach high precision (which is necessary for stability analysis). Another possibility is to invert equation (53) using a finite difference method or a finite element method [329]. This approach requires that boundary conditions can be obtained through equation (54). Finally,

it is worth noting that the inversion of Poisson's equation for rotating stars can be performed via the mapping technique, e.g. transformation of space coordinates [41,70]. To our knowledge, the mapping technique has never been used for discs.

Since the efficient numerical computation of the potential is not trivial, people still seek approximate semi-analytical approaches that would complement purely numerical solutions and make them faster. This is in particular the case for the infinite slab (IS) approximation, which was introduced by Paczyński and collaborators in the late 1970s in a series of papers devoted to marginally stable discs [186, 260, 261].[14] In this approach the disc is examined in (R, z)-plane, imposing axial symmetry and omitting the radial and azimuthal terms in equation (53). The Poisson equation then reads

$$\frac{\partial^2 \nu_{\mathrm{d}}}{\partial z^2} = 4\pi\rho. \tag{55}$$

The main advantage of equation (55) is the possibility of determining the magnitude of the vertical gravity by direct integration along the vertical direction. In this way even a full solution $\nu_{\mathrm{d}}(R, z)$ could apparently be found. The vertical field is given by

$$g_z^{\mathrm{d}}(z) = -\frac{\partial \nu_{\mathrm{d}}}{\partial z} = -4\pi\Sigma(z), \tag{56}$$

where

$$\Sigma(R, z) = \int_0^z \rho(R, z')\,\mathrm{d}z' \tag{57}$$

is half of the local surface density. This result is exact if the system is invariant under any translation perpendicular to z-axis, and for this the disc must be (i) of infinite size, (ii) strictly flat (its geometrical thickness $2H$ is constant) and (iii) homogeneous in the radial direction (vertical gradients can be non-zero in the direction of z-axis). Obviously, such properties are not realistic, but the validity of the IS approximation can be extended to radially inhomogeneous, finite size discs with non-constant thickness, provided that the condition

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \nu_{\mathrm{d}}}{\partial R}\right) \ll \frac{\partial^2 \nu_{\mathrm{d}}}{\partial z^2} \tag{58}$$

is satisfied. It is not generally possible to check inequality (58) for a given configuration because the "true" potential ν_{d} is not known or is difficult to compute with sufficient accuracy (for the technical reasons mentioned above). Also, the above-described way of getting $\nu_{\mathrm{d}}(R, z)$ cannot be used to estimate the radial component of the field.

It is worth noting that equation (56) holds regardless of the relative masses of the disc and the central object, making the IS approximation often very useful (for example, it can be used to compute massive non-Keplerian discs), but it is also important to know where this powerful approximation fails. Although the potential is a global function, we can determine this by considering its local properties by expressing $\nu_{\mathrm{d}} \propto R^{a_1}|z|^{a_2}$. Inequality (58) is violated in regions where

$$\frac{|z|}{R} \gg \sqrt{\frac{a_2(a_2 - 1)}{a_1^2}}, \tag{59}$$

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[14] See also numerous follow-up papers which helped to clarify substantial uncertainties in the original formulation: [28, 40, 53, 151, 152, 154, 211, 265, 299, 316–319, 329]. In particular, the role of self-gravity and the disk fragmentation in fuelling the central BH has been discussed, but this topic remains beyond our discussion.

or, in terms of density gradients, when

$$\left(\nabla_R \rho\right)^2 \gg \left(\nabla_z \rho + z^{-1}\rho\right)\left(\nabla_z \rho + 2z^{-1}\rho\right). \tag{60}$$

For a reasonable density stratification and attractive potential, we expect $a_1 < 0$ and $1 < a_2 < 2$. This implies that the IS approximation may break down when (i) $a_1 \sim a_2$, which corresponds to discs and tori of moderate and large aspect ratios, and/or (ii) the radial and vertical density gradients are of the same order. This applies if density waves or shocks with steep fronts are present, especially in boundary layers or close to the disc edges. The flatter is the disc, the weaker are the edge effects.

An interesting property of vertical stratification can be derived directly from the IS approximation [119, 260]. If the disc is able to reach hydrostatic equilibrium, then the total pressure gradient compensates total gravity. Differentiating equation of hydrostatic balance relative to z gives

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{1}{\rho}\frac{\mathrm{d}P}{\mathrm{d}z} + \frac{Mz}{(R^2 + z^2)^{3/2}}\right] + 4\pi\rho = 0. \tag{61}$$

This equation becomes independent of the radius in the limit of a disc that stays in vertical equilibrium under its own gravity. An analytical solution can be obtained assuming polytropic relation $P = c_s^2 \rho^{1 + \frac{1}{n}}$ (c_s is a speed, n is polytropic index). The above-given equation is analogy of the famous hydrostatic equilibrium problem for gaseous spheres in equilibrium. The resulting "plane" Lane–Emden equation writes [123]

$$\frac{1}{\rho}\frac{\mathrm{d}}{\mathrm{d}z}\left(\rho^{\frac{1-n}{n}}\frac{\mathrm{d}\rho}{\mathrm{d}z}\right) + \frac{4\pi n}{c_s^2\left(1 + n\right)} = 0. \tag{62}$$

For vertically isothermal structure with $n \to \infty$, the equation (62) is the Frank–Kaménetski equation [303]

$$\frac{\mathrm{d}^2 \ln \rho}{\mathrm{d}z^2} + \frac{4\pi\rho n}{c_s^2\left(1 + n\right)} = 0 \tag{63}$$

and the solution is [123, 265, 299]

$$\rho(z) = \rho_c \operatorname{cosech}^2\left(zc_s^{-1}\sqrt{2\pi\rho_c}\right), \tag{64}$$

where c_s is the isothermal sound speed and $\rho_c \equiv \rho(0)$ is the mid-plane density. From equation (57), the surface density of the self-gravitating disc is given by $\Sigma = \sqrt{2P_c/\pi}$ [265], and thus the disc mass depends solely on the radial distribution of central pressure $P_c = \rho_c c_s^2$. Discussion of solutions for various polytropic indices can be found in [155, 260]. The issue is that, as the polytropic index n runs from infinity (in the isothermal case) to smaller values, the vertical density profile tends to be flatter. In the limit $n \to 0$, this profile becomes exactly rectangular in shape. Figure 4 shows the density profile in the direction perpendicular to the disc plane in a typical disc. The isothermal case is also shown for comparison. Models of the disc vertical stratification are close to the models of stellar interiors and show that the vertical structure is not isothermal and cannot be modelled by a polytrope (with $n = \text{const}$), at least not if the disc is optically thick. Discs are heated by turbulent motion and this results in a very complex vertical stratification of physical quantities.
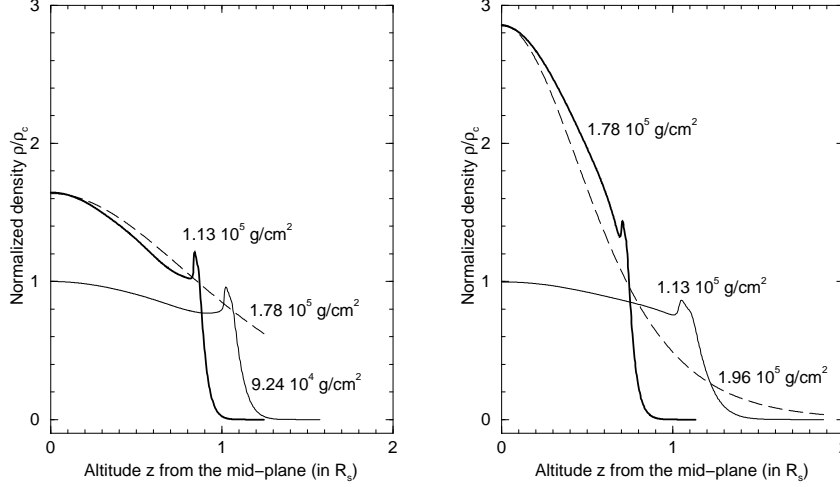
Figure 4. The vertical distribution of density in a standard-type disc (same conditions as for figure 3) at distance $r = 500 r_{\mathrm{S}}$ (left panel) and $r = 600 r_{\mathrm{S}}$ (right panel). Three cases are shown, all with $M_{\bullet} = 10^8 M_{\odot}$: (i) In the first case (thin solid line), hydrostatic equilibrium is maintained by the central object while the disc self-gravity is deliberately neglected. The mid-plane density is then equal to ρ_{c}. These calculations include vertical heat transfer by radiation and convection. (ii) In the second case (thick solid line), the disc vertical self-gravity is introduced in the IS approximation [152]. Despite the evident accumulation of matter closer to the disc mid-plane, the area under the curves (which corresponds to half the surface density $\Sigma(H)$, as labelled on each curve) is almost unchanged. Details of stratification are sensitive to the mode of energy dissipation. (iii) Also shown is the solution for a self-gravitating disc in the approximation of the Lane-Emden equation with isothermal structure (as given by equation (64); dashed line).

4.3. What criterion for self-gravity in thin discs?

In many respects it would be useful for the disc modelling if the degree of self-gravity were indicated by an unmistakable criterion. As a relevant quantity, one can intuitively suggest the mass ratio $q \ (= M_{\mathrm{d}}/M_{\bullet})$, where

$$M_{\mathrm{d}}(R) = 4\pi \int^{R} \Sigma(R, H) R' \, \mathrm{d}R' \tag{65}$$

is the cumulative mass of the disc and $\Sigma(H)$ is a half of the local surface density according to equation (57). One might postulate that self-gravity is unimportant for $q \lesssim 10\%$, say. Note that, in a binary system, the q-value (defined as the mass ratio of the two stars) is critical in the definition of the Roche lobes. However, such a criterion is unreliable to explore an extended disc in its close neighbourhood, where it cannot be reduced to a point mass. We show below that the right criterion is based on the quantity $\sim qR/H$ which contains information about the disc mass (relative to the central black hole) as well as about the distribution of the mass density in space (in terms of the aspect ratio).

The ratio $\zeta = g_z^{\mathrm{d}}/g_z^{\bullet}$ can be evaluated provided that the vertical component of the disc self-gravity is known. For $\zeta = 1$, the disc and the central body contribute comparably to the field along z-axis (this case is referred as the "weak self-gravitating

limit", WSG hereafter). As soon as ζ exceeds unity, the disc is *vertically* self-gravitating. Within the IS approximation, $g_z^{\rm d}$ is given by equation (56), so one obtains $\zeta = 4\pi\Sigma(R,z)r^3/(M_\bullet z)$. When the gas remains confined within altitudes $z/R \ll 1$, we can use the mid-plane value

$$\zeta_{\rm c} \sim \frac{4\pi\rho_{\rm c}R^3}{M_\bullet}, \tag{66}$$

where $\rho_{\rm c}$ is the mass density in $z = 0$, as defined in the absence of self-gravity. The role of parameter $\zeta(R,z)$ was discussed in [151]; Taylor expansion around $z = 0$ shows that the mid-plane value of ζ can be identified with A-parameter of Paczyński [260]. One finds that the upper limit for the disc height is typically $H \lesssim 0.3R$; such discs stand on a borderline beyond which the approximation of geometrically thin discs becomes very crude and inadequate. We notice that vertical self-gravity occurs preferentially (i) when the central BH has a small mass, (ii) if the local mass density in the disc is large, and (iii) at rather large distances from the centre. If the surface density obeys a power-law in radius, for instance in the form $\Sigma = \Sigma_0 R^s$, then

$$\zeta_{\rm c} = q(R)\,(s+2)\left(\frac{H}{R}\right)^{-1} \qquad (s \neq -2). \tag{67}$$

Relation (67) shows that even low-mass discs can be self-gravitating if they are flat enough. We can put numbers in this relation: given that standard discs in active galaxies have a power index $s \sim -0.6$ and small, almost constant aspect ratio, $H/R \sim 0.01$, we find $\zeta_{\rm c} = 140 \times q(R)$ [151]. (The adopted value of s corresponds to the outer, gas-pressure dominated region; another value would not change qualitative conclusions at this point.) In other words, a disc becomes vertically self-gravitating when its relative mass exceeds a small threshold value. This value is model dependent; typically it is order of a few $\times 10^{-2}$. Note that the limit is much less than the crude estimate of 10% suggested before.

By introducing ratio $\eta \equiv \nabla_R\nu_{\rm d}/\nabla_R\nu_\bullet$ one can follow the same kind of analysis for the radial component of gravity. A disc is *radially* self-gravitating if $|\eta| > 1$, hence this regime can be referred to as a strongly self-gravitating (SSG) disc. Conclusions are less straightforward in this case because we lack sufficiently reliable expression for $g_R \equiv -\nabla\nu_{\rm d}$. In other words, there is no obvious analogy to the IS approximation. Some authors [236, 346] have proposed that the total potential can be approximated by the monopole term, $\nu_{\rm d} \sim -M_{\rm d}(R)/R$, which is not fully satisfactory but gives at least the right order of magnitude in the SSG limit ($\eta \sim 1$). Considering, as above, a power law for the radial surface density profile, η is given by

$$\eta_{\rm c} = q(R) - \zeta_{\rm c}\frac{H}{R} = (1+s)q(R) \tag{68}$$

in the equatorial plane. Using equation (67), we find

$$\eta_{\rm c} = \zeta_{\rm c}\frac{1+s}{2+s}\frac{H}{R} \tag{69}$$

(the case of $s = -2$ can be derived in similar manner). The SSG regime corresponds to

$$\zeta_{\rm c} = \frac{2+s}{1+s}\left(\frac{H}{R}\right)^{-1} \gg 1. \tag{70}$$

Using the above values for s and H/R, $\eta_{\rm c} = 1$ corresponds to $\zeta_{\rm c} \sim 150$, well above the WSG limit. As $\zeta_{\rm c}$ is expected to be an increasing function of radius, the SSG limit

involves more mass and is reached at larger radii than the weak limit. It therefore lies inside the vertically self-gravitating discs. This value is in good agreement with the results established by Lantian & Xiaoci [190] who have attempted to performed a direct, analytical integration of equation (54). Although their investigation still contains some rough approximations (for instance, singularities in the Poisson kernel were artificially removed), these authors derived a formula for radial self-gravity which accounts for finite size effects. For $s = -1$, they found radial acceleration to be

$$g_R^{\mathrm{d}} = 2\Sigma \left[\ln \frac{(2R+H)(R+R_{\mathrm{in}})(R_{\mathrm{out}}-R)}{(2R-H)(R-R_{\mathrm{in}})(R_{\mathrm{out}}+R)} - \frac{H^2}{4R^2} \right], \tag{71}$$

which sets the SSG limit at $\zeta_{\mathrm{c}} \sim 67$ (assuming the Eddington accretion rate; see figure 3 right). Note that this formula still remains singular at the disc edge [234]. To circumvent the difficulty, a softening parameter is often introduced [195], however, correct behaviour can be obtained just by careful treatment of edges.

Results discussed so far have not yet addressed the essential questions concerning the gravitational stability of self-gravitating, geometrically thin discs. It is known for long, and confirmed by observations, that galaxies (treated as discs of stars, gas and dust) are regulated by self-gravity [38, 342]. Similar processes can form spirals and bar patterns in purely gaseous discs. It was shown in a remarkable paper by Goldreich & Lynden-Bell [123] that instability sets in when

$$\frac{\pi \rho_{\mathrm{c}}}{4\Omega^2} \gtrsim \xi_n, \tag{72}$$

where the lower limit, ξ_n, depends on polytropic index n (the value of $\xi_n = 1.1$ corresponds to a vertically isothermal disc, while $\xi_n = 1.75$ for $n \to 0$). Note that this is an analogy of Toomre's original criterion, but with a slightly different threshold [120, 128, 278, 341]. It is important to recall that Goldreich & Lynden-Bell's criterion has been established from an analysis of growing unstable modes through a linear perturbation theory, within the framework of IS approximation and assuming a non-viscous, uniformly rotating, flat disc. It is therefore expected to be somewhat altered when relaxing some of the above assumptions. It is easy to recognise that equation (72) also gives $\zeta_{\mathrm{c}} \gtrsim 18$–$28$.

We thus conclude that instabilities are not present in a disc if its size remains below the weak limit. The region separating the weak, the unstable and the strong limit can be narrow or wide, depending on the surface density profile; or, more precisely, on the function $\zeta_{\mathrm{c}}(R)$. Again, a very nice example is provided by Saturn's rings which are vertically self-gravitating at their outer edge, but gravitationally stable, implying that the weak limit and the unstable limit do not coincide.

4.4. The self-gravitating regime in accretion discs in active galaxies

According to equation (66), an increase of the ζ-parameter is expected as the radius increases. This means that there is a boundary zone that separates the inner (non-self-gravitating) solution from the outer (self-gravitating) one. The location of this zone can be found without accounting for self-gravity, simply by computing ζ from a disc model [78, 128]. This approach may underestimate the magnitude of self-gravity and overestimate the radius of such boundary, but it provides a useful order of magnitude estimate. A more reliable result must include vertical self-gravity into consideration. This can be achieved within the IS approximation. Furthermore, in the context of

α-prescription, the WSG limit is reached at $R_{\rm sg} \equiv R(\zeta = 1)$. For accretion rates lower than a certain critical value, $\dot{M}_{\rm crit}$, one finds [151]

$$R_{\rm sg} \sim 480 \, \bar{\alpha}^{14/27} \dot{M}^{-8/27} M_8^{-2/3} \bar{\kappa}^{2/9} \mu^{-8/9}, \tag{73}$$

while

$$R_{\rm sg} \sim 230 \, \bar{\alpha}^{2/9} \dot{m}^{4/9} M_8^{-2/3} \bar{\kappa}^{2/3} \tag{74}$$

for accretion rates larger than $\dot{M}_{\rm crit}$; here, $R_{\rm sg}$ is expressed in Schwarzschild radii, and the critical accretion rate is expressed in solar masses per year, $\dot{M}_{\rm crit} \sim 0.27 \, \alpha_{0.1}^{2/5} \bar{\kappa}^{-3/5} \mu^{-6/5} \, M_\odot/{\rm yr}$ ($\bar{\kappa}$ is opacity). Noticeable deviations are observed when considering temperature- and density-dependent opacities and equation of state (not yet specified self-consistently in equations (73)–(74)). The existence of two different relations arises from the fact that BH discs generally possess an inner region (dominated by radiation pressure) and an outer (gas pressure dominated) region where the physical quantities (namely, the density) exhibit different variations with radius and the accretion rate. The case (74) corresponds approximately to accretion luminosity exceeding the Eddington limit.

Figure 5 gives a theoretical relationship between the location of WSG limit and the accretion rate. It was computed by different authors using the α-prescription and the same input parameters (relevant for accretion flows onto super-massive BHs in galactic nuclei). It is noticeable that no firm agreement has been achieved about the position of this limit, which largely fluctuates from one author to another simply because of different ingredients in models. Also, there are no direct observations available to confirm if this WSG-transition operates in nature and if it occurs where predicted. However, in active galaxies there is a *striking similarity* between the location and size of the so-called broad-line region, and the value of $R_{\rm sg}$. This correspondence could indicate that the clouds are formed at the distance where self-gravity becomes important (cf. [75] for a systematic discussion of the correlation). According to bi-dimensional calculations, a lower (reliable) limit appears to be $R_{\rm sg} \sim 530 r_{\rm S}$ for a 10^8 M$_\odot$ central black hole and $\alpha = 0.1$. This is only 5 milliparsecs from the centre, much less than previously estimated (e.g. [316]).

4.5. One dimensional models: self-similar and asymptotic solutions

Of simple models that can be constructed analytically on the basis of α-parameterisation, we pay a special attention to those where self-gravity is taken into account in the vertical direction, while Keplerian rotation is still imposed. This represents even a weaker category of WSG discs, because their rotation curve is assumed not to be altered by the disc mass. Within this approach, the above-discussed infinite slab approximation has been found very convenient, although some criticism can be raised at this point, because these disc models typically tend to be rather massive. Hence they become globally self-gravitating and no more Keplerian (this problem occurs especially if the surface density falls faster than the square of radius). Paczyński's original paper [260] was devoted to the investigation of discs with constant ratio of the vertical self-gravity to the vertical component of the central field, i.e. constant value of ζ-parameter (i.e. $\zeta = g_z^{\rm d}/g_z^\bullet$, which corresponds to A in Paczyński's notation). The assumption of vertical self-gravity is justified by the fact that strongly self-gravitating discs are prone to gravitational instabilities [123, 341] which should make them evolve in a couple of dynamical time scales ($t_{\rm dyn} \sim \Omega^{-1}$). The endpoint
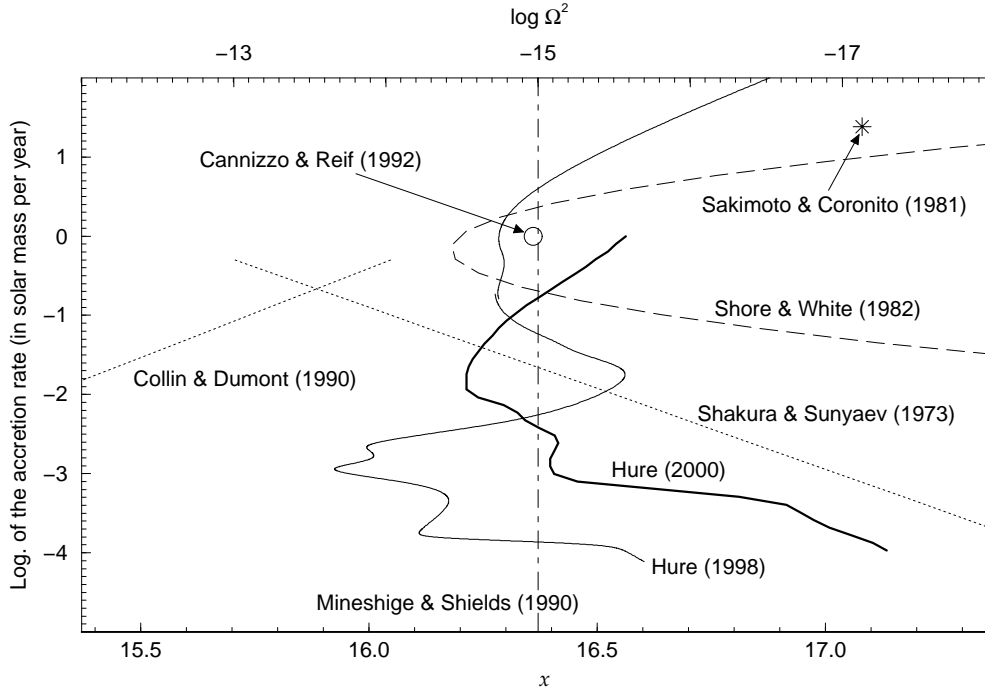
Figure 5. The location of WSG limit as a function of the accretion rate in an α-disc ($\alpha = 0.1$). Top axis gives the logarithm of Keplerian orbital frequency (Ω in Hz), while the bottom axis gives the corresponding value of $x \equiv \log R - \frac{1}{3}\log M_8$ (radius is in cm, the mass M_8 is in units of 10^8 solar masses). The accretion rate is on the vertical axis. Individual curves correspond to different models as indicated. The WSG limit was computed without including vertical self-gravity in all models except Huré (1998, reference [151]; and 2000, reference [152]). The case [151] is one dimensional while [152] is two-dimensional. Distortions in the curves are given mainly by opacity which shows drastic variations with temperature. A reference power-law line for the standard Shakura & Syunaev (1973, [314]) non-self-gravitating disc is also plotted.

of this evolution is the state of constant Toomre's Q-parameter, which is typical for differentially rotating discs of stars (Q can be related to ζ via equation (66)).

Soon it was realized that holding $\zeta = $ const might produce some substantial bias [319]; a series of papers followed to elucidate the local response of the disc to vertical self-gravity and to study $\zeta(R)$ profile. Keplerian discs were investigated in which ζ varies as a function of radius in a self-consistent fashion (another motivation for this research was the chance of discovering new branches of solutions). Most explorations were carried out for steady discs, using the so-called "vertically averaged approximation", where one seeks only for radial profiles of density, temperature and other physical quantities (without considering vertical stratification in detail). With additional assumptions about opacity, equation of state, pressure regime, etc., one is naturally lead to self-similar solutions [231, 236, 346].

As already mentioned above, vertical self-gravity makes the accreted gas more concentrated around the mid-plane. As a result, the disc tends to collapse in the vertical direction beyond some critical radius where $\zeta \gtrsim 1$ (cf. figures 3 and 4). This

Model	T_c	Σ	H	ζ_c
α-disc (no self-gravity)	$-9/10$	$-6/10$	$21/20$	$27/20$
Self-gravitating α-disc	0	3	$-3/8$	9
β-disc (no self-gravity)	$-7/8$	$-1/2$	$17/16$	$23/16$
Self-gravitating β-disc	$-7/8$	$-1/2$	$-3/8$	$23/8$
Q-constant disc	$-9/7$	$-15/7$	$6/7$	0
Irradiated Q-constant disc	$-5/8$	$-29/16$	$19/16$	0

Table 1. In the WSG regime, parameters of the disc are found proportional R^s in the radial direction. In this table, the values of power-law index, s, are given for different physical quantities: mid-plane temperature $T_c(R)$, surface density $\Sigma(R)$, geometrical thickness $H(R)$, and self-gravity parameter $\zeta_c(R)$. The optically thick limit was assumed and the α-, β- and Q-constant prescriptions were adopted for turbulent viscosity. Self-similarity is maintained if gas pressure dominates in the medium with opacity coefficient and chemical composition being constant [28, 154]. Viscosity was incorporated e.g. by [236] in a self-similar time-dependent model. In spite of numerous papers, it would be still premature to expect a general consensus regarding the viscosity model. For further details, see [40, 53, 119, 139, 186, 211, 299, 319, 329].

property is shared by all computations of geometrically thin discs, independently of the actual viscosity prescription (in other words, if one assumes α-parameterisation for viscosity, then the disc height is almost independent of α). The opposite is true for geometrically thick discs which tend to be thicker when self-gravity is taken into account [40]. If prescription for viscosity is specified, the variation with radius can be found in explicit form. Examples are given in table 1; given the viscosity law and the heating mode of the disc that is on verge of self-gravitating regime, one finds the R-variation of the disc thickness, temperature, surface density, etc. Note that $\zeta(R)$ increases with R in all models. This leads inevitably to the onset of self-gravity, subsequent reduction of the disc thickness and further consequences on many aspects, such as self-irradiation, dynamical stability and the emitted spectrum of the disc. It may also affect the disc warping [281] in regions that are shadowed from the central source and cannot reprocess its radiation; although weak at those distances, the resulting lack of heating is in favour of gravitational instability.

Properties derived through the α-prescription deserve a comment. The R-dependence of the ζ comes out rather steep and much more pronounced than what is obtained for a bare standard disc (neglecting self-gravity), in which case $\zeta_c(R) \propto R^{27/20}$. This means that the transition from non-self-gravitating regime to the fully self-gravitating one should occur in a very narrow region; the weak, unstable and strong limits are expected to almost coincide with each other. As a consequence of the steep rise of self-gravity, the disc can suffer from a runaway-type increase of density; one is even led to densities typical for stellar interiors and conditions suitable for deuterium and hydrogen burning [265]. Temperature reaches a plateau in the outer region of the disc plane with value [93, 128, 152, 316]

$$T = \mu \, m_H \, k^{-1} \left(\tfrac{2}{3} \, \alpha^{-1} \dot{M} \right)^{2/3} \simeq 2.4 \times 10^4 \, \mu \, \alpha_{0.1}^{-2/3} \, \dot{m}^{2/3} \quad [K], \qquad (75)$$

where μm_H is the mean mass per particle, $\dot{m}$ is the accretion rate normalized to $0.1 M_\odot/\mathrm{yr}$ and $\alpha_{0.1} = \alpha/0.1$. Some authors have argued that such uniform mid-plane temperatures are unphysical and they have suggested the use of a different prescription for viscosity [28, 93]. Interestingly, this may lead back to the original idea of Paczyński that viscosity is connected with the Q-parameter.
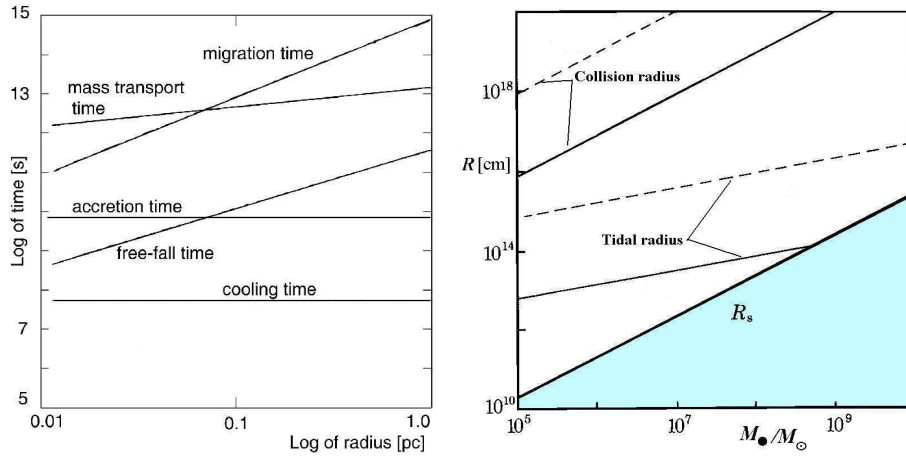
Figure 6. Left: time-scales in the gravitationally unstable region of an accretion disc [76]. One can define the mass transport time in the disc, cooling time (relevant for the formation of fragments), accretion time (for a star growing to a pre-determined mass, $M_\star \sim 10 M_\odot$), migration time (for a star sinking towards the centre), and free-fall time. Right: length-scales for a black hole embedded in a star cluster: the tidal disruption radius within which stars are disrupted, the collision radius within which colliding stars are disrupted, and Schwarzschild radius (R_s) of the central black hole. Solar-type stars (solid lines) and solar-mass red giants (dashed lines) have been assumed. Figure was adapted from [101, 114].

We conclude this section by showing several typical scales of the problem in figure 6. These graphs are derived from basic arguments and they can help to identify physical mechanisms which are important from the observational point of view. Time-scales relevant for the gravitationally unstable regions of accretion discs are shown as a function of radius (in the left frame). The possibility of star formation was taken into account together with continuous migration and evolution in the medium (here, $M_\bullet = 10^6 M_\odot$ was assumed for the central black hole). If the stars form a dense (quasi-spherical) nuclear cluster in which the central BH resides, one can define the radius where stars are destroyed tidally or by their mutual collisions. The resulting value depends on $M_\bullet$ as well as details of the stellar structure (as shown in the right frame).

5. Waiting for observations and constraints

Significant effort has been devoted to searching for black holes during the last few decades. Traditionally, most approaches of present-day astronomy rely on the electromagnetic radiation coming from cosmic sources, but there are other possibilities which will be surely exploited in near future (neutrinos, high-energy cosmic particles, and gravitational waves). The idea of detecting black holes via (electromagnetic) radiation of nearby luminous matter can be traced back to the celebrated lecture, which rev. John Michell presented in front of the Royal Society as early as in 1783.[15]

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[15] A paper was based on that lecture and published under the title *On the means of discovering the distance, magnitude, &c. of the fixed stars, in consequence of the diminution of the velocity of their light...* in Philosophical Transactions of the Royal Society of London, 1784 **74** 35.
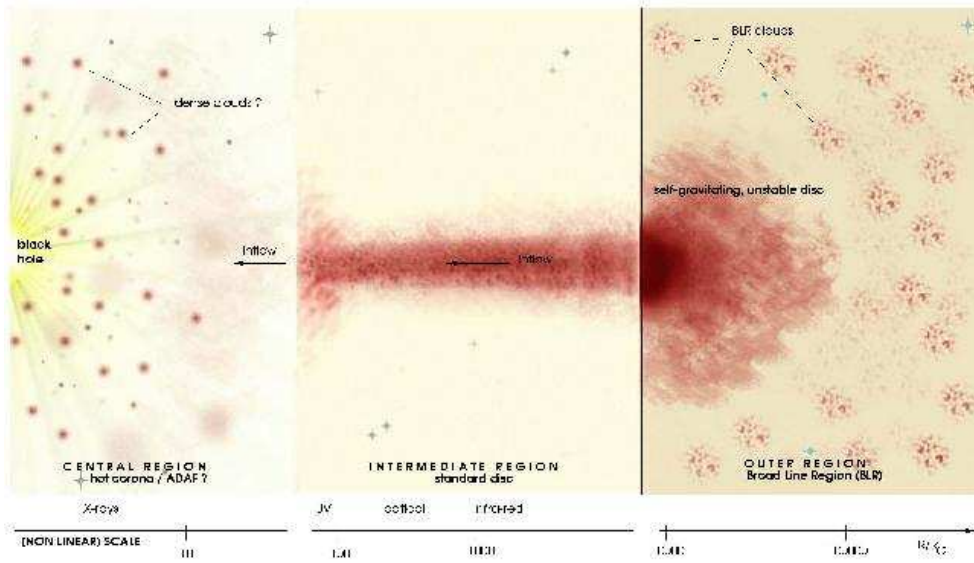
Figure 7. A schematic drawing of the central part of a galactic nucleus. Different regions of the accretion flow are shown: the innermost, radiation-pressure dominated region with a central BH, the intermediate region with a standard disc, and the outer, gravitationally unstable part. Length-scales are proportional to the central mass. The indicated values are valid for $M_\bullet = 10^8 M_\odot$ and slightly model-dependent [75].

Understanding local physics of the medium is crucial if one aims to interpret complex and variable spectra that are observed. X rays are particularly pertinent because they are produced near BH, as discussed by Fabian and others [101]. Indeed, given the intrinsic emissivity of the medium, there are only a few factors determining the source spectrum on the Earth. The main factor is the black hole mass $M_\bullet$; the angular momentum of the black hole and the disc self-gravity play secondary roles. The geometrical optics approximation is thus adequate and one can hope to infer actual values of BH parameters from the motion of test matter around it. However, one has to be well aware of various environmental influences that hinder real observations.

Under conditions that are thought to be typical in active galactic nuclei, the inner regions of accretion flows contain diluted plasma with a corona threaded by magnetic fields [199,279]. High-energy photons are first generated in the corona. These primary photons irradiate substantially colder gas of the accretion disc, where they are reprocessed further (figure 7). A fraction of photons is reflected. The temperature of the reflecting medium can be $\sim 10^4$ K in galactic nuclei and about 2–3 orders of magnitude higher in the case of Galactic BH candidates [56,92]. The quality of photon reprocessing depends critically on the ionization state and other local properties of the medium. In this way the resulting spectrum is formed with spectral features (lines) superposed on top of the background continuum. X-ray spectra contain precious information about rapid flows from which one can infer properties of the central source of gravity and the state of the nearby gas.

Manifestation of strong gravity has been searched in radiation of accretion discs with the main aim of finding the evidence for the central black hole [102,223,292,293].

Majority of these papers deal with non-self-gravitating discs and assume that integrated light of the whole source is only very weakly affected by the disc gravitation (however, see [164, 322, 348]). In the case of our Galactic centre, the presence of standard-type (thin) accretion disc is rather improbable (e.g. Narayan [245]). Direct effect of the disc gravity on spectral line profiles is below current observational resolution for any present-day spectra of active galactic nuclei, so it can be safely neglected in the first approximation (we briefly discuss this effect in the following paragraph). It is noteworthy, however, that self-gravity of the disc has indirect impact on spectral features. First, as discussed in section 4, the disc geometrical shape is affected by self-gravity with consequences for obscuration of some parts of the disc and the resulting variability, especially at large values of inclination (this is also the case when GR effects of the central BH are maximal, so the two influences are mixed). Self-gravity also controls the amount of disc self-irradiation, which produces indirect signs in reflected light. Finally, self-gravity induces fragmentation in the disc medium (depending in particular on the disc cooling properties [120]) with fundamental consequences on the disc structure. Fragments can even lead to star formation or contribute to star-bursts [76, 96]. The clumpy material may also result in much more pronounced variability observed from the source [8].

Let us now discuss direct effects of the combined BH–disc gravity on observed light rays. In the early works of Cunningham & Bardeen [83, 84], a standard non-self-gravitating disc was assumed. These authors initiated a series of investigations of model spectra with the aim of revealing the imprints of GR. Since then, many people have advanced various modifications of the original semi-analytical approach [24, 48, 105, 224, 353]. Very prominent role has been played by studies of spectral features (lines) of iron in the X-ray band; in particular, the so called disc-line scheme has been widely applied, as it captures the main spectral characteristics attributed to strong gravity [103, 369]. The line is produced by reprocessing the primary continuum radiation near above the inner edge of the disc, within several tens of r_s (see [247, 297] for recent exposition of emission mechanisms). The line intrinsic width is small, typically much less than the final observed profile that results from integration over the whole emitting surface. Standard picture of the disc thermal spectrum [257, 264] was reconsidered to account also for the disc gravity [192]. For detailed examination it would be useful to produce templates of expected line profiles or even spectral models that are consistent with the formation of non-axially symmetric structures in accretion discs as well as the presence of a black hole in the centre [138, 165, 348]. There is an exciting possibility of detecting gravitational effects on polarized light of a source orbiting a black hole, but this idea still awaits technology usable for its practical implementation [79, 230]. Effects of the black hole itself on the light propagation are blurred in real sources in which the signal comes from different, insufficiently resolved regions.

So far, only a few cases have been discovered of an accreting black hole with a line that attains the characteristic shape, width and variability and could be understood as the interplay of Doppler and gravitational shifts. The best example of such spectral feature originating near a BH has been identified in the nucleus of Seyfert galaxy MCG–6-30-15[16] where the iron line was caught at the moment of high redshift [104, 161]. Large gravitational redshift can be connected with rapid BH rotation, because the radius of the disc inner edge (where the line presumably originates) decreases with BH

[16] Cf. Morphological Catalogue of Galaxies.

angular momentum increasing [362]. Magnetic fields can change this straightforward conclusion. Also, the line is strongly variable, so its final (continuum subtracted) shape cannot be resolved reliably with current knowledge; but there is a good prospect of overpassing these deficiencies with more detailed models and using time-resolved spectroscopy that will be able to track rapid features [233]. Reconstruction of accretion disc images will then be doable using reverberation techniques [274, 327] or Doppler mapping [222]. This will also require good understanding of non-standard effects which are usually neglected, e.g. the effect of disc eccentricity [23, 326].

As far as BH candidates in the Milky Way are concerned, the relativistic iron line was detected in a well-known micro-quasar GRS 1915+105 [225]; rapid rotation of the central black hole is not necessary in order to explain the data in this case. Similar evidence for the broad and skewed iron line has been discovered with different degree of credibility in about eight other sources [226]. In spite of a limited similarity of the relativistic iron line in these sources, we recall that recall that accretion discs have negligible gravity in BH candidates, and so we turn back to accretion discs galactic centres.

Moving to a region upstream the accretion flow — to the distance of hundreds r_s and farther out from the core of a galaxy, the interaction between gaseous flows and other astronomical bodies (stars forming a nuclear cluster) can substantially change the structure, accretion rate and chemical composition of the flow, so we briefly touch this point, too. That way, the medium feeding the BH can be enriched above the level of usual solar metalicity [21, 64, 74, 77]. On the other hand, moving satellites feel the presence of the interstellar gas, especially in nuclei of active galaxies with high concentrations of stars and the relatively dense interstellar environment [31]. Accreting medium thus influences the motion and evolution of these bodies. Even though self-gravitational effects of the medium become progressively less important near the centre, a relatively accurate knowledge of their impact is desired in order to derive, reliably, BH parameters from rotational motion of the surrounding luminous matter. The dissipative interaction of stellar-mass satellites and the resulting orbital decay proceed with a time-scale that is defined by properties of both components of the system: in the inner disc, hydrodynamical drag dominates the orbit evolution, while in the self-gravitating outer region, the disc structure becomes clumpy and direct collisional interaction is reduced. In even more remote parts, gravitational relaxation is important for the cluster evolution, so the outer cluster is dynamically distinct from the core.

Syer, Clarke & Rees [332] proposed that a stellar-mass body passing the galactic core can be captured on a quasi-stable bound orbit around a central super-massive BH. In the simplest version of the model, stars have been considered as bullets crashing on the disc surface and passing through at hyper-sonic speed. Zurek, Siemiginowska & Colgate [375], Armitage, Zurek & Davies [20] and Ivanov, Igumenshchev & Novikov [160] examined various effects of passages (of individual stars) through diluted gaseous environment (slab-disc geometry was assumed; other people explored different contexts of this problem [64, 166, 255, 256]). Pineault & Landry [276] discussed the statistical rate and distribution of impacts over the disc surface. Self-gravity of the disc medium was also considered [128, 317, 357], as well as GR effects in stellar motion [168, 356], effects of the orbital decay of satellites by gravitational radiation and close encounters [137, 244, 287, 331], or stellar coalescences [72, 116, 288, 290]. Gradual changes of the physical and chemical state of satellites and of the disc itself were discussed [21]. Radiation torques [259] as well as star–gas hydrodynamic interaction [125, 136, 207]
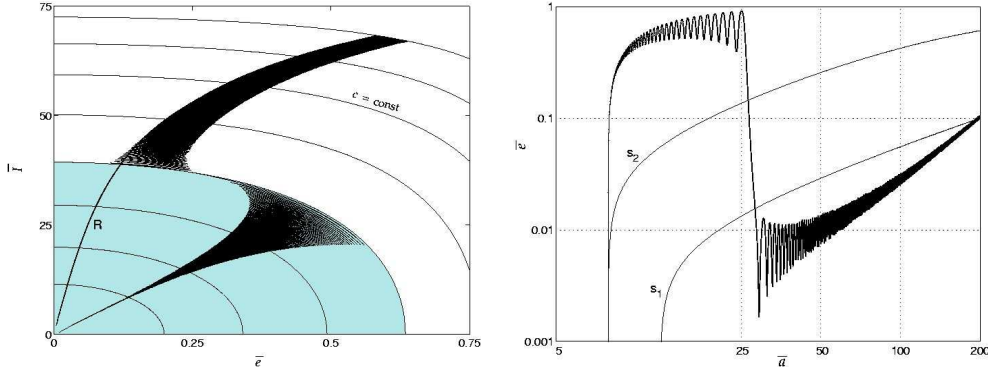
Figure 8. Left: Mean inclination $\bar{I}$ vs. eccentricity $\bar{e}$ of a satellite orbit that crosses a gravitating disc. Contour lines of the Kozai parameter, $c = $ const, are also plotted. The orbital evolution consists of oscillations along c-contours and slow diffusion across them, towards the origin $\bar{e} = \bar{I} = 0$. (This example has $\bar{e}_0 = 0.6$, $\bar{I}_0 = 70°$ as initial condition.) Kozai's quasi-integral is not isolating here, because of the orbital decay via dissipative transitions through the disc medium. An approximate analytic solution is also shown by a monotonic curve R, neglecting the influence of the disc gravity [289]. Right: Mean eccentricity $\bar{e}$ versus semi-major axis $\bar{a}$ of a star crossing a gravitating disc. The orbit is represented by an eccentricity curve with oscillations overlying the long-term decay. Also shown are two reference orbits having the same initial parameters (the solid curve s_1) or the same final parameters (s_2) as a simplified situation without the disc gravity (the latter two curves exhibit continuous and purely monotonic circularisation). Large excursions in eccentricity occur if the disc mass is non-zero, albeit it can be much less than the central mass (here, $M_d \sim 10^{-3} M_\bullet$). Figure adopted from [357].

induce different channels of angular momentum transport in the medium. The clumpy structure [118, 189] of the outer, self-gravitating disc reduces the efficiency of direct star–disc collisions, while turbulence in the medium tends to strengthen dissipation [375].

Repetitive transitions bring the satellites into the plane of the disc at almost circular orbits [289, 332] (though, one can speculate of several effects that slow the process of circularisation down, so that eccentric or non-equatorial orbits may be maintained even at late stages [124, 145, 354]). Subsequent orbit evolution is reduced to a situation that was addressed in connection with the formation and migration of bodies inside proto-planetary discs [248, 359, 367]. The inner cluster becomes flattened and its structural dimension is related to the size of the disc. One can immediately recognise plethora of consequences connected with non-sphericity, anisotropy and satellite segregation in the cluster: namely, the initial mass function is modified, as well as the form of spectral-line profiles and stellar velocity dispersions. The onset of star formation leads to gradual conversion of gas into stars in the disc [76, 77, 141].

Star-burst phenomenon has been indeed identified in some active galactic nuclei on $\sim 10^2$ pc scales, which could represent outskirts of a dusty torus surrounding the central black hole [18, 302]. Indeed, violent star formation could coincide with the outer parts of a massive nuclear cluster that is embedded within the dusty torus. Even if $M_d \ll M_\bullet$, the effect of the disc gravity on the long-term evolution of stellar orbits should be taken into account together with dissipation via star–disc (hydrodynamical) interactions. This stands in contrast with the fact that gravitation of accretion discs

has been neglected in most works exploring the nuclear cluster structure. The two influences are linked through their dependence on the disc density. A remark is worth making regarding the mass ratio q (which was criticised as unsuitable self-gravity criterion in section 4.3): it is a useful measure to assess the importance of stellar transitions across a self-gravitating disc in galactic nuclei, because attraction of the disc and the effects of transitions are both increasing, roughly proportionally, with M_d.

Circularisation of the orbits, evolution of their inclination and the resulting capture rate are visibly affected by the disc gravity. An important point is the range of time-scales: (i) dynamical period (of satellite stars revolution around the central mass; the shortest time-scale), (ii) period of oscillations in orbital elements (medium time-scale; these oscillations are due to disc gravity) and (iii) time for bringing the orbital plane into the disc (caused by successive interactions with the disc; long time-scale). The orbital decay continues also when satellites are embedded in the disc plane; its rate varies depending on the ability of the body to clear a gap along its trajectory through the medium. Different modes depend on the satellite masses, sizes and on the disc thickness and surface density at corresponding radius. The evolution of orbital parameters is an issue in extra-solar planetary research [272], which tackles similar systems on much small scales — cf. the discussion of planetary formation and migration [19, 145, 208, 248]. These topics have not yet been pursued to much detail in the context of galactic nuclear clusters embedded in the gaseous environment.

One can understand the origin of orbital parameters oscillations in terms of Kozai's theory [184], taking into account the gradual adiabatic diffusion of quasi-integrals of motion. In order to see the reason for orbital oscillations, one borrows from classical mechanics' traditional approach to the disc gravitational potential $\nu_d(\boldsymbol{r})$ as a perturbation acting on the satellite motion [49, 200, 240]. Again, the total potential is supposed to be dominated by the central field of the core, while the disc contribution ν_d can be transformed by a canonical transformation to the form which depends on mean orbital elements (eccentricity $\bar{e}$, inclination $\bar{I}$, and argument of pericentre $\bar{\omega}$; the bar denotes mean values) and does *not* depend on the fast-changing variable (mean anomaly). See [357] for the explicit form of this new $\langle\nu_d\rangle$ in terms of integration (over the mean anomaly) along an unperturbed Keplerian ellipse; in that paper, Kuzmin's potential was employed as a simple model for the disc gravitational field (it would be quite interesting to examine a generalization in terms of Miyamoto & Nagai's potential [239], which can be still performed analytically). In the first order of perturbation theory one finds that $\langle\nu_d\rangle \sim \mathrm{const} \equiv \langle\nu_d\rangle_0$; the mean semi-major axis $\bar{a}$ is also constant and differs from the actual osculating semi-major axis by short-period terms only. The mentioned averaging procedure allows us to disregard short-term effects and concentrate on the long-term evolution. Furthermore, owing to the axial symmetry of $\nu_d(R, z)$ exists an additional constant of motion [184]: $\sqrt{1 - \bar{e}^2}\cos\bar{I} = c \equiv \mathrm{const}$ (figure 8 left). This way the problem of orbital evolution is reduced to the evolution of $\bar{e}$ and $\bar{\omega}$ which are further constrained by condition $\langle\nu_d\rangle(\bar{e}, \bar{\omega}; c, \bar{a}) = \langle\nu_d\rangle_0$. Contours of the perturbation potential $\langle\nu_d\rangle$ provide a convenient representation of mean orbital elements in the $(\bar{e}, \bar{\omega})$-plane. Finally, *with the inclusion of orbital dissipation in the gaseous medium, one finds that an adiabatic drift occurs and drives the orbit across the contours of constant $\langle\nu_d\rangle$; subsequently, the drift provokes excursions of the orbital elements when a separatrix is traversed.* A representative example of a trajectory is shown in figure 8 (right). The interplay between Kozai's mechanism and gradual changes of the perturbation potential can give occasional (but large) eccentricity
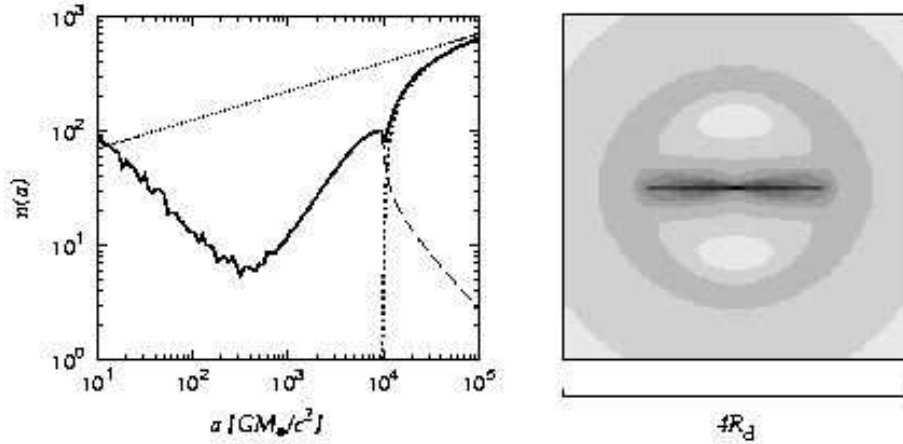
Figure 9. Structure of a nuclear cluster in interaction with a planar, gravitating disc. In this model [167,330], the long-term evolution of stellar orbits is influenced by dissipative transitions across the disc slab and by (small) gravitational attraction of the disc itself. Left: Number density $n_*(a)$ of satellites within a shell of semi-major axis $\langle a, a + \mathrm{d}a \rangle$ (solid curve). Two populations can be distinguished, corresponding to the satellites of the outer cluster (dotted) and to the highly flattened inner cluster (dashed). Thin dotted power-law line indicates the equilibrium structure of the Bahcall-Wolf cluster ($n \propto a^{1/4}$). Right: The number density $n_*(a)$ is represented by different levels of shading here. An azimuthal section of the cluster is plotted. In both plots, the gradual concentration of satellites towards the equatorial plane and formation of a ring structure are evident.

jumps; one of them is clearly visible in the figure.

Next step would be to consider a consistent scheme for the whole cluster of satellites. In their classical paper, Bahcall & Wolf [22] anticipated a way to reveal a hypothetical central massive hole (embedded in a collisional environment of a globular cluster) through stellar velocity dispersion and broadened wings of line profiles integrated over the BH domain of influence. In order to calculate the form of central cusps, the authors employed time-dependent diffusion equation. Of other assumptions, the most important was the spherical symmetry and isotropy of the cluster itself. Probability density of satellites can be expressed in terms of their semi-major axes: $n_*(a) \propto a^{1/4}$, where $n_*(a)\,\mathrm{d}a$ denotes the number of satellites in a shell with radius a and width $\mathrm{d}a$. Corresponding spatial density $n_*(r)$ is thus proportional to $r^{-7/4}$. Modifying the model assumptions (namely, that about the importance of collisions) leads to different but similar density profiles (power-law at large radii). For example, the well-known Plummer [277] sphere has a finite-density core with density falling off as r^{-5}, while Jaffe [162] and Hernquist [142] models both decline $\propto r^{-4}$ at large radii. The latter case has a sound theoretical basis in the mechanics of violent relaxation. See also [175,270,315,368] and, for more recent discussion and references, see [11,333]. Further, the collisional steady-state solutions have been reconsidered as a model for a cusp in the center of a galaxy [99]. Conclusions of these papers can be compared with more advanced models that relax some of the assumptions and include new effects, e.g. resonances, enhanced loss-cone depletion or rotation of the cluster [114,173,286,291,373]. Nevertheless, self-consistent treatment of non-spherical

stellar systems with the effects of self-gravity *and* the presence of an accretion disc are still an open problem. The disc induced dissipative orbital decay is distinct in its tendency to flatten the stellar component and even form a ring of stars that could be resolved observationally.

Figure 9 illustrates a stationary density profile of the cluster in a model where its members revolve around a BH and their orbits slowly decay due to dissipative effects. One can distinguish different populations of satellites (see the left panel): those which form almost spherical cluster far from the centre, outside the disc outer radius, and those which have been dragged into the disc plane near the centre (the latter form a ring). Hence it is a certain generalisation of the Bahcall–Wolf [22] cluster. A complementary representation is provided in the right panel of figure 9 which shows an azimuthal section across the cluster. Again, the inner cluster is evidently flattened and its dimension is tightly related to the size of the disc. It should be recalled that, with this approach, a relatively small volume is explored near BH, where the central mass has direct dynamical influence on the embedded stars and gas (in other words, rotational support of the stellar motion is not a negligible fraction compared to their random motions). The linear size of this region is a function of $M_\bullet$ and $\sigma_\star$, and is typically less than $\sim 10\,\mathrm{pc}$.

Note that the $\log M_\bullet \sim 4 \log \sigma_\star$ relation resembles the empirical correlation which has been determined in a large sample of galactic bulges [108, 121] (although the constant of proportionality can be different in the innermost region around the central BH, which is unresolved by current observations). Hence, in the inner cluster of stars, the obtained velocity dispersion should be determined by the joint action of gravity and of other agents, including dissipation in the interstellar environment. It was proposed [167, 330] that the effect of interaction between stars and the accretion flow shapes the observed $M_\bullet$–$\sigma_\star$ relation in the innermost region of some active galaxies (those with sufficiently dense accretion discs in the core). The model of star–disc interaction leads to an axially symmetric, but non-spherical central cluster, and so the exact value of the power-law index of the $M_\bullet$–$\sigma_\star$ relation depends also on the observer line of sight (the expected central velocity dispersion comes out somewhat larger for systems seen edge-on than for pole-on view). The model therefore represents rather extreme case, however, it is obvious that the interaction between stars and the intergalactic environment should be taken into account also at larger distances from the core [366]. Miralda-Escudé & Kollmeier [238] applied a similar idea to a much wider region (and hence, applicable to actual present-day observations). These authors propose star-disc collisions are a self-regulating process that helps to feed the central BH and control its growth. The influence of the collisions remains imprinted in the bulge mass–stellar velocity dispersion relation, which concerns the region in galactic nuclei greatly exceeding the domain of direct gravitational influence of the super-massive BH.

Naively, one could expect that dissipation of satellite's orbital energy and momentum should always lead to a very dense nuclear cluster. However, the outcome may be more complicated because of non-sphericity and satellite segregation (caused by differing pace of orbital evolution, according to the mass function and the distribution of sizes of stellar bodies). For example, recently, Zier & Biermann have showed [374], on the basis of numerical simulations, that a flattened shape of the cluster may arise due purely to gravitational effect of a massive binary BH in the core. They found that a stellar ring is formed on a parsec scale in the orbital plane of the central binary, while the cluster is gradually depleted. Another approach to

(self-gravitating, eccentric) galactic stellar discs was applied to the double nucleus of M 31 [300] and even the case of Milky Way was discussed recently [205]. Indeed, as indicated also in this paper, there is a broad variety of reasons why the compact nuclear cluster should acquire a non-spherical structure in the course of evolution [194, 302]. The effects exhibited by self-gravitating accretion flows are merely one ingredient of a complex problem.

6. Conclusions

Gravitational field of black-hole discs induces great diversity of effects and gives ample scope for new ideas, of which we could touch only very limited section. First of all, as noted in the first part of this paper, even some very basic questions remain still open within the framework of exact general relativistic solutions for the black hole–disc gravitational field. Our interest in global gravitational effects connected with astrophysical discs has lead us to focus on systems around super-massive BHs that reside in nuclei of galaxies, in which case the onset of self-gravity appears inevitable. Recent ideas of heavy tori being formed in certain moments of neutron-star binary coalescence make these topics fitting also to stellar mass objects, although their "discs" must be very different in many respects. Because of expected instabilities, accreting BHs are pertinent also for forthcoming gravitational wave experiments.

Astrophysical black holes are supposed to be embedded in complex cosmic environment which consists of multiple sub-systems spanning enormous range in the parameter space of masses, lengths and time-scales. Gaseous discs represent just one of the components in this formation. Under such circumstances, analytical means can offer insight into physical mechanisms which are connected with the disc, but specific analytical solutions lack generality and must be inevitably simplified. In recent decades, much effort has been devoted to clarify, which processes dominate the evolution and what are the applicable ranges of model parameters. It is quite possible that some regimes of the disc operation have been missed in previous works.

In this respect, an attractive recent speculation suggests that intermediate mass black holes (with $M_\bullet \sim 10^3 M_\odot$) might also exist, possibly in cores of some globular clusters. If this new family of black holes is revealed and if they also accrete gas from their close environs (current observational evidence is rather scarce), it would provide a fascinating opportunity to study accreting black hole sources for which the dynamical time is of the order of ~ 1 second. In other words, the expected variability of such object could span over intervals that can be readily sampled over many periods with available technology.

We concentrated our attention on a fairly conventional case of gas discs around super-massive black holes at small, sub-parsec length-scales where the motion of stellar-mass orbiters can serve as an accurate tool to probe the system. This idea requires various intervening influences (such as the perturbation of the central gravitational field caused by the disc/torus own gravity, or the orbital dissipation in the gaseous interstellar environment) to be understood with sufficient accuracy. Interesting and fundamental questions, namely, whether and where exactly the onset of self-gravity occurs in real accretion discs, still remain open to further research.

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References

[1] Abramowicz M A, Arkuszewski W and Muchotrzeb B 1976 Pressureless disk cannot be the source of the Kerr metric *Lett. Nuovo Cim.* **15** 476

[2] Abramowicz M A, Calvani M and Nobili L 1983 Runaway instability in accretion disks orbiting black holes *Nature* **302** 597

[3] Abramowicz M A, Curir A, Schwarzenberg-Czerny A and Wilson R E 1984 Self-gravity and global structure of accretion discs *Mon. Not. R. Astron. Soc.* **208** 279

[4] Abramowicz A, Henderson F and Ghosh P 1983 Evolution of accretion tori orbiting black holes. I – Theory *Mon. Not. R. Astron. Soc.* **203** 323

[5] Abramowicz M, Jaroszynski M and Sikora M 1978 Relativistic accreting disks *Astron. Astrophys.* **63** 221

[6] Abramowicz M A, Karas V and Lanza A 1998 On the runaway instability of relativistic tori *Astron. Astrophys.* **331** 1143

[7] Abramowicz M A and Percival M J 1997 Thermal equilibria of accretion discs around black holes: a review of recent progress *Class. Quantum Grav.* **14** 2003

[8] Abrassart A and Czerny B 2000 Toy model of obscurational variability in active galatic nuclei *Astron. Astrophys.* **356** 475

[9] Alekseev G A 2001 New integral equation form of integrable reductions of Einstein equations *Theor. Math. Phys.* **129** 1466

[10] Alekseev G A and Garcia A A 1996 Schwarzschild black hole immersed in a homogeneous electromagnetic field *Phys. Rev. D* **53** 1853

[11] Alexander T 1999 The distribution of stars near the supermassive black hole in the Galactic center *Astrophys. J.* **527** 835

[12] Amendt P, Lanza A and Abramowicz M A 1989 Baroclinic toroidal configurations of equilibrium *Astrophys. J.* **343** 437

[13] Andalib S W, Tohline J E and Christodoulou D M 1997 A survey of the principal modes of nonaxisymmetric instability in self-gravitating accretion disk models *Astrophys. J. Suppl.* **108** 471

[14] Ansorg M 2001 Differentially rotating disks of dust: Arbitrary rotation law *Gen. Rel. Grav.* **33** 309

[15] Ansorg M., Kleinwächter A and Meinel R 2003 Uniformly rotating axisymmetric fluid configurations bifurcating from highly flattened Maclaurin spheroids *Mon. Not. R. Astron. Soc.* **339** 515

[16] Ansorg M, Kleinwächter A and Meinel R 2003 Relativistic Dyson rings and their black hole limit *Astrophys. J.* **582** L87

[17] Ansorg M, Kleinwächter A, Meinel R and Neugebauer G 2002 Dirichlet boundary value problems of the Ernst equation *Phys. Rev. D* **65** 044006

[18] Aretxaga I 2003 Line formation in the inner starburst regions of AGN, in *Winds, Bubbles and Explosions* ed Arthur J and Henney W *Rev. Mex. Astron. Astrophys.* **15** 329

[19] Armitage P J, Livio M, Lubow S H and Pringle J E 2002 Predictions for the frequency and orbital radii of massive extrasolar planets *Mon. Not. R. Astron. Soc.* **334** 248

[20] Armitage P J, Zurek W H and Davies M B 1996 Red giant–disk encounters: food for quasars? *Astrophys. J.* **470** 237

[21] Artymowicz P, Lin D N C and Wampler E J 1993 Star trapping and metallicity enrichment in quasars and active galactic nuclei *Astrophys. J.* **409** 592

[22] Bahcall J N and Wolf R A 1976 Star distribution around a massive black hole in a globular cluster *Astrophys. J.* **209** 214

[23] Bao G, Hadrava P and Østgaard E 1996 Line profiles from relativistic eccentric rings *Astrophys. J.* **464** 684

[24] Bao G, Wiita P J and Hadrava P 1998 General relativistic effects on the spectrum reflected by accretion disks around black holes *Astrophys. J.* **504** 58

[25] Barbour J and Pfister H (ed) 1995 *Mach's principle – From Newton's Bucket to Quantum Gravity* (Boston: Birkhauser)

[26] Bardeen J M 1973 Timelike and null geodesics in the Kerr metric, in *Black Holes* ed DeWitt C and DeWitt B S (New York: Gordon and Breach) p 215

[27] Bardeen J M 1973 Rapidly rotating stars, disks, and black holes, in *Black Holes* ed DeWitt C and DeWitt B S (New York: Gordon and Breach) p 241

[28] Bardou A, Heyvaerts J and Duschl W 1998 Influence of viscosity laws on the transition to the self-gravitating part of accretion disks *Astron. Astrophys.* **337** 966

[29] Barrabès C and Israel W 1991 Thin shells in general relativity and cosmology: The lightlike limit *Phys. Rev. D* **43** 1129

[30] Baxandall F E, Newall H F and Stratton F J M 1930 The spectrum of β Lyrae *Ann. Solar Phys. Obs. Cambridge* **2** 1

[31] Begelman M C and Rees M J 1978 The fate of dense stellar systems *Mon. Not. R. Astron. Soc.* **185** 847

[32] Beig R and Schmidt B 2000 Time-independent gravitational fields, in *Einstein's Field Equations and Their Physical Implications*, Lect. Notes Phys. 540, ed Schmidt B G (Berlin: Springer) p 325

[33] Bičák J 1979 On the theories of the interacting perturbations of the Reissner-Nordström black hole *Czech. J. Phys. B* **29** 945

[34] Bičák J 2000 Selected solutions of Einstein's field equations: their role in general relativity and astrophysics, in *Einstein's Field Equations and Their Physical Implications*, Lect. Notes Phys. 540, ed Schmidt B G (Berlin: Springer) p 1

[35] Bičák J and Ledvinka T 1993 Relativistic disks as sources of the Kerr metric *Phys. Rev. Lett.* **71** 1669

[36] Bičák J and Ledvinka T 2000 Electromagnetic fields around black holes and Meissner effect *Nuovo Cim. B* **115** 739

[37] Bičák J, Lynden-Bell D and Katz J 1993 Relativistic disks as sources of static vacuum spacetimes *Phys. Rev. D* **47** 4334

[38] Binney J and Tremaine S 1987 *Galactic dynamics* (Princeton: Princeton Univ. Press)

[39] Blaes O M 1987 Stabilization of non-axisymmetric instabilities in a rotating flow by accretion on to a central black hole *Mon. Not. R. Astron. Soc.* **227** 975

[40] Bodo G and Curir A 1992 Models of self-gravitating accretion disks *Astron. Astrophys.* **253** 318

[41] Bonazzola S, Gourgoulhon E and Marck J-A 1998 Numerical approach for high precision 3D relativistic star models *Phys. Rev. D* **58** 104020

[42] Bondi H 1952 On spherically symmetrical accretion *Mon. Not. R. Astron. Soc.* **112** 195

[43] Bonnor W B 1992 Physical interpretation of vacuum solutions of einstein's equations. Part I. Time-independent solutions *Gen. Rel. Grav.* **24** 551

[44] Bonnor W B, Griffiths J B and MacCallum M A H 1994 Physical interpretation of vacuum solutions of Einstein's equations. Part II. Time-dependent solutions *Gen. Rel. Grav.* **26** 687

[45] Boyarchuk A A, Bisikalo D V, Kuznetsov O A and Chechetkin V M 2002 *Mass Transfer in Close Binary Stars*, Advances in Astronomy and Astrophysics 6 (London: Taylor & Francis)

[46] Boyer R H 1965 Rotating fluid masses in general relativity *Proc. Cambridge Phil. Soc.* **61** 527

[47] Bretón N, García A A, Manko V S and Denisova T E 1998 Arbitrarily deformed Kerr-Newman black hole in an external gravitational field *Phys. Rev. D* **57** 3382

[48] Bromley B C, Chen K and Miller W A 1997 Line emission from an accretion disk around a rotating black hole: Toward a measurement of frame dragging *Astrophys. J.* **475** 57

[49] Brower D and Clemence G 1961 *Methods of Celestial Mechanics* (New York: Academic Press)

[50] Brown G E, Lee C-H, Wijers R A M J, Lee H K, Israelian G and Bethe H A 2000 A theory of gamma-ray bursts *New Astronomy* **5** 191

[51] Butterworth E M and Ipser J R 1976 On the structure and stability of rapidly rotating fluid bodies in general relativity. I. The numerical method for computing structure and its application to uniformly rotating homogeneous bodies *Astrophys. J.* **204** 200

[52] Campanelli M and Lousto C O 1999 Second order invariant gravitational perturbations of a Kerr black hole *Phys. Rev. D* **59** 124022

[53] Cannizzo J K and Reiff C M 1992 Accretion disks in active galactic nuclei – Vertically averaged models *Astrophys. J.* **385** 87

[54] Carter B 1973 Black hole equilibrium states, in *Black Holes* ed DeWitt C and DeWitt B S (New York: Gordon and Breach) p 57

[55] Carter B 1979 The general theory of the mechanical, electromagnetic and thermodynamic properties of black holes, in *General Relativity, An Einstein Centenary Survey* ed Hawking S W and Israel W (Cambridge: Cambridge Univ. Press) p 294

[56] Castro-Tirado A J, Greiner J and Paredes J M (ed) 2001 *Microquasars*, Astrophys. Space Sci. Rev. 276 (Dordrecht: Kluwer)

[57] Chakrabarti S K 1988 Spacetime with self-gravitating thick disc *J. Astrophys. Astron.* **9** 49

[58] Chandrasekhar S 1969 *Ellipsoidal figures of equilibrium* (New Haven: Yale University Press)

[59] Chandrasekhar S 1978 The Kerr metric and stationary axisymmetric gravitational fields *Proc. R. Soc. London A* **358** 405

[60] Chandrasekhar S 1979 An introduction to the theory of the Kerr metric and its perturbations, in *General Relativity, An Einstein Centenary Survey* ed Hawking S W and Israel W (Cambridge: Cambridge Univ. Press) p 370

[61] Chandrasekhar S 2000 *The Mathematical Theory of Black Holes* (New York: Oxford Univ. Press)

[62] Chaudhuri S and Das K C 1997 Two-soliton solutions of axially symmetric metrics *Gen. Rel. Grav.* **29** 75

[63] Chaudhuri S and Das K C 1997 Axially symmetric metrics from Laplace's seed by inverse scattering method *J. Math. Phys.* **38** 5792

[64] Cheng K S and Wang J-M 1999 *Astrophys. J.* **521** 502

[65] Chinea F J and Gonzáles-Romero L M (ed) 1993 *Rotating Objects and Relativistic Physics*, Lect. Notes Phys. 423 (Berlin: Springer)

[66] Chrzanowski P L 1975 Vector potential and metric perturbations of a rotating black hole *Phys. Rev. D* **11** 2042

[67] Chrzanowski P L 1976 Applications of metric perturbations of a rotating black hole: Distortion of the event horizon *Phys. Rev. D* **13** 806

[68] Ciufolini I and Wheeler J A 1995 *Gravitation and Inertia* (New Jersey: Princeton Univ. Press)

[69] Clarke C J S 1993 *The Analysis of Space-Time Singularities*, Cambridge Lect. Notes Phys. 1 (Cambridge: Cambridge Univ. Press)

[70] Clément M J 1974 On the solution of the Poisson equation for rapidly rotating stars *Astrophys. J.* **194** 709

[71] Clément G 2000 Generating rotating Einstein-Maxwell fields *Annalen Phys.* **9** SI-42

[72] Colgate S A 1967 Stellar coalescence and the multiple supernova interpretation of quasi-stellar sources *Astrophys. J.* **150** 163

[73] Collin S, Boisson C, Mouchet M, Dumont A-M, Coupé S *et al* 2002 Are quasars accreting at super-Eddington rates? Astrn. Astrophys. **388** 771

[74] Collin S and Huré J-M 1999 Influence of the metallicity and of the irradiation on the structure of accretion disks around massive black holes *Astron. Astrophys.* **341** 385

[75] Collin S and Huré J-M 2001 Size-mass-luminosity relations in AGN and the role of the accretion disc *Astron. Astrophys.* **372** 50

[76] Collin S and Zahn J 1999 Star formation and evolution in accretion disks around massive black holes *Astron. Astrophys.* **344** 433

[77] Collin S and Zahn J 1999 Star formation in accretion disks around massive black holes and pregalactic enrichment *Astrophys. Space Sci.* **265** 501

[78] Collin-Souffrin S and Dumont A-M 1990 Line and continuum emission from the outer regions of accretion discs in active galactic nuclei. II – Radial structure of the disc *Astron. Astrophys.* **229** 292

[79] Connors P A, Stark R F and Piran T 1980 Polarization features of X-ray radiation emitted near black holes *Astrophys. J.* **235** 224

[80] Cosgrove C M 1980 Relations between the group-theoretic and soliton-theoretic techniques for generating stationary axisymmetric gravitational solutions *J. Math. Phys.* **21** 2417

[81] Cosgrove C M 1981 Bäcklund transformations in the Hauser-Ernst formalism for stationary axisymmetric spacetimes *J. Math. Phys.* **22** 2624

[82] Cosgrove C M 1982 Relationship between the inverse scattering techniques of Belinskii-Zakharov and Hauser-Ernst in general relativity *J. Math. Phys.* **23** 615

[83] Cunningham C T 1975 The effects of redshifts and focusing on the spectrum of an accretion disk around a Kerr black hole *Astrophys. J.* **202** 788

[84] Cunningham C T and Bardeen J M 1972 The Optical appearance of a star orbiting an extreme

Kerr black hole *Astrophys. J.* **173** L137

[85] Czerny B 2003 Comparison of GBH and AGN accretion flows, in *Active Galactic Nuclei: From Central Engine to Host Galaxy*, ASP Conf. Series 290, ed Collin S, Combes F and Shlosman I (San Francisco: Astronomical Society of the Pacific) p 59

[86] Daigne F and Mochkovitch R 1997 Gamma-ray bursts and the runaway instability of thick discs around black holes *Mon. Not. R. Astron. Soc.* **285** L15

[87] Damour T, Soffel M and Xu C 1991 General-relativistic celestial mechanics. I. Method and definition of reference systems *Phys. Rev. D* **43** 3273

[88] Damour T, Soffel M and Xu C 1992 General-relativistic celestial mechanics. II. Translational equations of motion *Phys. Rev. D* **45** 1017

[89] de Andrade V C, Blanchet L and Faye G 2001 Third post-Newtonian dynamics of compact binaries: Noetherian conserved quantities and equivalence between the harmonic-coordinate and ADM-Hamiltonian formalisms *Class. Quantum Grav.* **18** 753

[90] Demianski M 1976 Stationary axially symmetric perturbations of a rotating black hole *Gen. Rel. Grav.* **7** 551

[91] De Villiers J-P and Hawley J F 2003 Global general relativistic magnetohydrodynamic simulations of accretion tori *Astrophys. J.* **592** 1060

[92] Done C 2002 Accretion flows in X-ray binaries *Phil. Trans. R. Soc. (Ser. A)* **360** 1967

[93] Duschl W J, Stittmatter P A and Biermann P L 2000 A note on hydrodynamic viscosity and selfgravitation in accretion disks *Astron. Astrophys.* **357** 1123

[94] Dyson F W 1893 The potential of an anchor ring *Phil. Trans. R. Soc. (Ser. A)* **184** 43

[95] Ellis G F R and Schmidt B G 1977 Singular space-times *Gen. Rel. Grav.* **8** 915

[96] Elmegreen B 2002 Star formation from galaxies to globules *Astrophys. J.* **577** 206

[97] Embacher F 1984 The analog of electric and magnetic fields in stationary gravitational systems *Found. Phys.* **14** 721

[98] Ernst F J 1968 New formulation of the axially symmetric gravitational field problem *Phys. Rev. D* **167** 1175; 1968 *ibid* **168** 1415

[99] Evans N W and Collett J L 1997 On the origin of cusps in stellar systems *Astrophys. J.* **480** L103

[100] Evans N W and de Zeeuw P T 1992 Potential-density pairs for flat galaxies *Mon. Not. R. Astron. Soc.* **257** 152

[101] Fabian A C 1979 Theories of the nuclei of active galaxies *Proc. R. Soc. London A* **366** 449

[102] Fabian A C, Iwasawa K, Reynolds C S and Young A J 2000 Broad iron lines in active galactic nuclei *Publ. Astron. Soc. Pacific* **112** 1145

[103] Fabian A C, Rees M J, Stella L and White N E 1989 X-ray fluorescence from the inner disc in Cygnus X-1 *Mon. Not. R. Astron. Soc.* **238** 729

[104] Fabian A C, Vaughan S, Nandra K, Iwasawa K, Ballantyne D R *et al* 2002 A long hard look at MCG–6-30-15 with XMM-Newton *Mon. Not. R. Astron. Soc.* **335** L1

[105] Fanton C, Calvani M, de Felice F and Cadez A 1997 Detecting accretion disks in active galactic nuclei *Publ. Astron. Soc. Japan* **49** 159

[106] Fernandes J F Q and Lun A W C 1996 Gauge invariant perturbations of black holes. I. Schwarzschild space-time *J. Math. Phys.* **37** 836

[107] Fernandes J F Q and Lun A W C 1997 Gauge invariant perturbations of black holes. I. Kerr space-time *J. Math. Phys.* **38** 330

[108] Ferrarese L and Merritt D 2000 A fundamental relation between supermassive black holes and their host galaxies *Astrophys. J.* **539** L9

[109] Fishbone L G and Moncrief V 1976 Relativistic fluid disks in orbit around Kerr black holes *Astrophys. J.* **207** 962

[110] Font J A and Daigne F 2002 The runaway instability of thick discs around black holes — I. The constant angular momentum case *Mon. Not. R. Astron. Soc.* **334** 383

[111] Font J A and Daigne F 2003 On the stability of thick accretion disks around black holes *Astrophys. J.* **581** L23

[112] Font J A and Ibáñez J M 1998 Non-axisymmetric relativistic Bondi-Hoyle accretion onto a Schwarzschild black hole *Mon. Not. R. Astron. Soc.* **298** 835

[113] Frank J, King A and Raine D 2002 *Accretion Power in Astrophysics* (Cambridge: Cambridge Univ. Press)

[114] Frank J and Rees M J 1976 Effects of massive central black holes on dense stellar systems *Mon. Not. R. Astron. Soc.* **176** 633

[115] Frauendiener J and Klein C 2001 Exact relativistic treatment of stationary counterrotating dust disks: physical properties *Phys. Rev. D* **63** 084025

[116] Freitag M and Benz W 2002 A new Monte Carlo code for star cluster simulations. II. Central

black hole and stellar collisions *Astron. Astrophys.* **394** 345

[117] Frolov V P and Novikov I D 1998 *Physics of Black Holes. Basic Concepts and New Developments* (Dordrecht: Kluwer)

[118] Fukuda H, Habe A and Wada K 2000 The effect of the self-gravity of gas on gas fueling in a barred galaxy with a supermassive black hole *Astrophys. J.* **529** 109

[119] Fukue J and Sakamoto C 1992 Vertical structures of self-gravitating gaseous disks around a central object *Publ. Astron. Soc. Japan* **44** 553

[120] Gammie C F 2001 Nonlinear outcome of gravitational instability in cooling, gaseous disks *Astrophys. J.* **553** 174

[121] Gebhardt K, Bender R, Bower G, Dressler A, Faber S M *et al* 2000 A relationship between nuclear black hole mass and galaxy velocity dispersion *Astrophys. J.* **539** L13; 2001 *ibid* **555** L75

[122] Gleiser R J, Nicasio C O, Price R H and Pullin J 2000 Gravitational radiation from Schwarzschild black holes: the second-order perturbation problem *Phys. Rep.* **325** 41

[123] Goldreich P and Lynden-Bell D 1965 Gravitational stability of uniformly rotating disks *Mon. Not. R. Astron. Soc.* **130** 97

[124] Goldreich P and Sari R 2002 Eccentricity evolution for planets in gaseous disks *Astrophys. J.* **585** 1024

[125] Goldreich P and Tremaine S 1980 Disk-satellite interactions *Astrophys. J.* **241** 425

[126] González G A and Letelier P S 2000 Rotating relativistic thin disks *Phys. Rev. D* **62** 064025

[127] Gonzáles G A and Letelier P S 2004 Exact general relativistic thick disks *Phys. Rev. D* in print

[128] Goodman J 2003 Self-gravity and quasi-stellar object discs *Mon. Not. R. Astron. Soc.* **339** 937

[129] Goodman J and Narayan R 1988 The stability of accretion tori. III – The effect of self-gravity *Mon. Not. R. Astron. Soc.* **231** 97

[130] Greenberg R and Brahic A 1984 *Planetary Rings* (Tuscon: Univ. Arizona Press)

[131] Gutsunayev Ts I and Manko V S 1985 On the gravitational field of a mass possessing a multipole moment *Gen. Rel. Grav.* **17** 1025

[132] Gutsunaev Ts I and Manko V S 1988 On a family of solutions of the Einstein-Maxwell equations *Gen. Rel. Grav.* **20** 327

[133] Hachisu I 1986 A versatile method for obtaining structures of rapidly rotating stars *Astrophys. J. Supp.* **61** 479; 1986 *ibid* **62** 461

[134] Hachisu I and Eriguchi Y 1983 Bifurcations and phase transitions of self-gravitating and uniformly rotating fluid *Mon. Not. R. Astron. Soc.* **204** 583

[135] Hachisu I, Eriguchi Y and Nomoto K 1986 Fate of merging double white dwarfs *Astrophys. J.* **308** 161

[136] Hagio F 1987 Angular momentum transport by star-gas interaction and structure of accretion disks *Publ. Astron. Soc. Japan* **39** 887

[137] Hameury J-M, King A R, Lasota J-P and Auvergne M 1994 Black-hole stellar accretion in active galactic nuclei *Astron. Astrophys.* **292** 404

[138] Hartnoll S A and Blackman E G 2002 Iron line profiles from black hole accretion discs with spiral velocity structure *Mon. Not. R. Astron. Soc.* **332** L1

[139] Hashimoto M, Eriguchi Y and Müller E 1995 Equilibrium structure of self-gravitating Keplerian disks *Astron. Astrophys.* **297** 135

[140] Hawley J F, Balbus S A and Winters W F 1999 Local hydrodynamic stability of accretion disks *Astrophys. J.* **518** 394

[141] Heller C H and Shlosman I 1994 Fueling nuclear activity in disk galaxies: starbursts and monsters *Astrophys. J.* **424** 84

[142] Hernquist L 1990 An analytical model for spherical galaxies and bulges *Astrophys. J.* **356** 359

[143] Hockney R W and Eastwood J W 1988 *Computer simulation using particles* (Bristol: Hilger Ed.)

[144] Hoenselaers C and Dietz W (ed) 1984 *Solutions of Einstein's Equations: Techniques and Results*, Lect. Notes Phys 205 (Berlin: Springer)

[145] Holman M, Touma J and Tremaine S 1997 Chaotic variations in the eccentricity of the planet orbiting 16 Cyg B *Nature* **386** 254

[146] Hoyle F and Lyttleton R A 1939 The evolution of stars *Proc. Camb. Phil. Soc.* **35** 592

[147] Hoyle F and Lyttleton R A 1941 On the accretion theory of stellar evolution *Mon. Not. R. Astron. Soc.* **101** 227 **35** 592

[148] Hubený I, Harmanec P and Shore S N 1994 On the strategy of future observations and modeling of the beta Lyr system *Astron. Astrophys.* **289** 411

[149] Hughes S A 2000 Evolution of circular, nonequatorial orbits of Kerr black holes due to gravitational radiation *Phys. Rev. D* **61** 084004; 2001 *ibid* **63** 064016; 2001 *ibid* **64** 064004

[150] Hunter C 1963 The structure and stability of self-gravitating disks *Astrophys. J.* **126** 299
[151] Huré J-M 1998 Properties of self-gravitating α-discs revisited. General scaling laws *Astron. Astrophys.* **337** 649
[152] Huré J-M 2000 On the transition to self-gravity in low mass AGN and YSO accretion discs *Astron. Astrophys.* **358** 378
[153] Huré J-M 2002 Origin of non-Keplerian motions of masers in NGC 1068 *Astron. Astrophys.* **395** L2
[154] Huré J M, Collin-Souffrin S, Le Bourlot J and Pineau des Forêts G 1994 Structure of the outer regions of accretion discs in Active galactic nuclei *Astron. Astrophys.* **290** 19
[155] Ibáñez M H and Sigalotti L 1984 The equilibrium of polytropic self-gravitating sheetlike masses *Astrophys. J.* **285** 784
[156] Igumenshchev I V, Chen Xingming and Abramowicz M A 1996 Accretion discs around black holes: two-dimensional, advection-cooled flows *Mon. Not. R. Astron. Soc.* **278** 236
[157] Igumenshchev I V, Narayan R and Abramowicz M A 2003 Three-dimensional magnetohydrodynamic simulations of radiatively inefficient accretion flows *Astrophys. J.* **592** 1042
[158] Islam J N 1985 *Rotating fields in general relativity* (Cambridge: Cambridge Univ. Press)
[159] Israel W 1966 Hypersurfaces and Thin Shells in General Relativity *Nuovo Cim. B* **44** 1
[160] Ivanov P B, Igumenshchev I V and Novikov I D 1998 Hydrodynamics of black hole-accretion disk collision *Astrophys. J.* **507** 131
[161] Iwasawa K, Fabian A C, Reynolds C S, Nandra K, Otani C *et al* 1996 The variable iron K emission line in MCG–6-30-15 *Mon. Not. R. Astron. Soc.* **282** 1038
[162] Jaffe W 1983 A simple model for the distribution of light in spherical galaxies *Mon. Not. R. Astron. Soc.* **202** 995
[163] Jantzen R T, Carini P and Bini D 1992 The many faces of gravitoelectromagnetism *Annals Phys.* **215** 1
[164] Karas V, Lanza A and Vokrouhlický D 1995 Emission-line profiles from self-gravitating thin disks *Astrophys. J.* **440** 108
[165] Karas V, Martocchia A and Šubr L 2001 Variable line profiles due to non-axisymmetric patterns in an accretion disc around a rotating black hole *Publ. Astron. Soc. Japan* **53** 189
[166] Karas V and Šubr L 2001 Orbital decay of satellites crossing an accretion disc *Astron. Astrophys.* **376** 686
[167] Karas V, Šubr L and Šlechta M 2002 Stellar dynamics in a galactic centre surrounded by an accretion disc, in *Gravitation: Following the Prague Inspiration* ed Semerák O, Podolský J and Žofka M (Singapore: World Scientific) p 85
[168] Karas V and Vokrouhlický D 1994 Relativistic precession of the orbit or a star near a supermassive black hole *Astrophys. J.* **422** 208
[169] Kato S 2001 Basic properties of thin-disk oscillations *Publ. Astron. Soc. Japan* **53** 1
[170] Kato S, Fukue J and Mineshige S 1998 *Black-hole accretion disks* (Kyoto: Kyoto Univ. Press)
[171] Kegeles L S and Cohen J M 1979 Constructive procedure for perturbations of spacetimes *Phys. Rev. D* **19** 1641
[172] Khanna R and Chakrabarti S K 1992 Effects of a self-gravitating disc on test particle motion around a Kerr black hole *Mon. Not. R. Astron. Soc.* **259** 1
[173] Kim E Einsel C Lee H M Spurzem R and Lee M G 2002 Dynamical evolution of rotating stellar systems – II. Post-collapse, equal-mass system *Mon. Not. R. Astron. Soc.* **334** 310
[174] King A R 2003 Accretion in compact binaries, in *Compact Stellar X-Ray Sources* ed Lewin W H G and van der Klis M (Cambridge: Cambridge Univ. Press) in print
[175] King I R 1966 The structure of star clusters. III. Some simple dynamical models *Astron. J.* **71** 64
[176] King M and Pfister H 2001 Electromagnetic Thirring problems *Phys. Rev. D* **63** 104004
[177] Klein C 1997 Counter-rotating dust rings around a static black hole *Class. Quantum Grav.* **14** 2267
[178] Klein C 2003 Exact relativistic treatment of stationary black-hole–disk systems *Phys. Rev. D* **68** 027501
[179] Klein C and Richter O 1998 Physically realistic solutions to the Ernst equation on hyperelliptic Riemann surfaces *Phys. Rev. D* **58** 124018
[180] Kopal Z 1978 *Dynamics of Close Binary Systems* Astrophys. Space Sci. Library 68 (Dordrecht: D Reidel)
[181] Kordas P 1999 Aspects of solution-generating techniques for space-times with two commuting Killing vectors *Gen. Rel. Grav.* **31** 1941
[182] Korotkin D 1997 Some remarks on finite-gap solutions of the Ernst equation *Phys. Lett. A* **229** 195

[183] Korotkin D and Matveev V 2000 *Funct. Anal. Appl.* **34** 1
[184] Kozai Y 1962 Secular perturbations of asteroids with high inclination and eccentricity *Astron. J.* **67** 591
[185] Kozłowski M, Jaroszynski M and Abramowicz M A 1978 The analytic theory of fluid disks orbiting the Kerr black hole *Astron. Astrophys.* **63** 209
[186] Kozłowski M, Wiita P J and Paczyński B 1979 Self-gravitating accretion disk models with realistic equations of state and opacities *Acta Astron.* **29** 157
[187] Krivan W and Price R H 1998 Initial data for superposed rotating black holes *Phys. Rev. D* **58** 104003
[188] Krolik J H 1999 *Active Galactic Nuclei* (Princeton: Princeton Univ. Press)
[189] Kumar P 1999 The structure of the central disk of NGC 1068: a clumpy disk model *Astrophys. J.* **519** 599
[190] Lantian Y and Xiaoci L 1990 Radial self-gravity of accretion disc around supermassive black holes *Astrophys. Space Sci.* **172** 293
[191] Lanza A 1992 Self-gravitating thin disks around rapidly rotating black holes *Astrophys. J.* **389** 141
[192] Laor A and Netzer H 1989 Massive thin accretion discs. I – Calculated spectra *Mon. Not. R. Astron. Soc.* **238** 897
[193] Larwood J D 1997 *The tidally induced warping, precession and truncation of accretion discs*, PhD Thesis (Queen Mary and Westfield College, London)
[194] Lauer T R, Gebhardt K, Richstone D, Tremaine S, Bender R *et al* 2002 Galaxies with a central minimum in stellar luminosity density *Astron J.* **124** 1975
[195] Laughlin G, Korchagin V and Adams F C 1998 The dynamics of heavy gaseous disks *Astrophys. J.* **504** 945
[196] Laughlin G and Różyczka M 1996 The effect of gravitational instabilities on protostellar disks *Astrophys. J.* **456** 279
[197] Ledvinka T 1998 *Thin disks as sources of stationary axisymmetric electrovacuum spacetimes*, PhD Thesis (Charles Univ., Prague)
[198] Lehner L 2001 Numerical relativity: a review *Class. Quantum Grav.* **18** R25
[199] Leighly K M 1999 A comprehensive spectral and variability study of narrow-line Seyfert 1 galaxies observed by ASCA *Astrophys. J. Supp.* **125** 297; 1999 *ibid* 125 317
[200] Lemaitre A and Dubru P 1991 Secular resonances in the primitive solar nebula *Celestial Mechanics and Dynamical Astronomy* **52** 57
[201] Lemos J P S and Letelier P S 1994 Exact general relativistic thin disks around black holes *Phys. Rev. D* **49** 5135
[202] Letelier P S 1989 On soliton solutions to the vacuum Einstein equations obtained from a general seed solution *Class. Quantum Grav.* **6** 875
[203] Letelier P S 1999 On the gravitational field of static and stationary axial symmetric bodies with multi-polar structure *Class. Quantum Grav.* **16** 1207
[204] Letelier P S and Oliveira S R 1987 Exact self-gravitating disks and rings: a solitonic approach *J. Math. Phys.* **28** 165
[205] Levin Y and Beloborodov A M 2003 Stellar disk in the Galactic center: a remnant of a dense accretion disk? *Astrophys. J.* **590** L33
[206] Levy H 1968 Classification of stationary axisymmetric gravitational fields *Nuovo Cim. B* **56** 253
[207] Lin D N C, Artymowicz P and Wampler E J 1994 Star-disk interaction in quasars and AGNs, in *Theory of Accretion Disks – 2* ed Duschl W J, Frank J, Meyer F, Meyer-Hofmeister E and Tscharnuter W M (Dordrecht: Kluwer)
[208] Lin D N C and Ida S 1997 On the origin of massive eccentric planets *Astrophys. J.* **477** 781
[209] Lin D N C and Papaloizou J C B 1996 Theory of accretion disks II: Application to observed systems *Ann. Rev. Astron. Astrophys.* **34** 703
[210] Linet B 1977 Stationary axisymmetric test fields on a Kerr metric *Phys. Lett. A* **60** 395
[211] Liu B F, Xie G Z and Ji K F 1994 Theoretical restrictions on accretion disks *Astrophys. Space Sci.* **220** 75; 1995 *ibid* **225** 299
[212] Longaretti P Y 2002 On the phenomenology of hydrodynamic shear turbulence *Astrophys. J.* **576** 587
[213] Lu Y, Cheng K S, Yang L T and Zhang L 2000 A massive thick disc around a massive black hole and its runaway instability *Mon. Not. R. Astron. Soc.* **314** 453
[214] Lynden-Bell D 1969 Galactic nuclei as collapsed old quasars *Nature* **223** 690
[215] Lynden-Bell D, Bičák J and Katz J 1999 On accelerated inertial frames in gravity and electromagnetism *Annals Phys.* **271** 1

[216] Lynden-Bell D, Katz J and Bičák J 1995 Mach's principle from the relativistic constraint equations *Mon. Not. R. Astron. Soc.* **272** 150

[217] Lynden-Bell D and Pringle J 1974 The evolution of viscous discs and the origin of the nebular variables *Mon. Not. R. Astron. Soc.* **168** 603

[218] MacFadyen A I and Woosley S E 1999 Collapsars: gamma-ray bursts and explosions in "failed supernovae" *Astrophys. J.* **524** 262

[219] Maison D 2000 Duality and hidden symmetries in gravitational theories, in *Einstein's Field Equations and Their Physical Implications*, Lect. Notes Phys. 540, ed Schmidt B G (Berlin: Springer) p 273

[220] Manko V S 1992 The exterior gravitational field of a static and stationary mass with an arbitrary set of multipole moments *Gen. Rel. Grav.* **24** 35

[221] Manko V S and Novikov I D 1992 Generalizations of the Kerr and Kerr-Newman metrics possessing an arbitrary set of mass-multipole moments *Class. Quantum Grav.* **9** 2477

[222] Marsh T R and Horne K 1988 Images of accretion discs. II – Doppler tomography *Mon. Not. R. Astron. Soc.* **235** 269

[223] Martocchia A 2000 *X-Ray spectral signatures of accreting black holes*, PhD Thesis (SISSA, Trieste)

[224] Martocchia A, Karas V and Matt G 2000 Effects of Kerr space-time on spectral features from X-ray illuminated accretion discs *Mon. Not. R. Astron. Soc.* **312** 817

[225] Martocchia A, Matt G, Karas V, Belloni T and Feroci M 2002 Evidence for a relativistic iron line in GRS 1915+105 *Astron. Astrophys.* **387** 215

[226] Martocchia A, Matt G, Karas V, Belloni T and Feroci M 2003 Iron line diagnostics for the GRS 1915+105 black hole, in *The Restless High Energy Universe*, Proc. of the 2nd BeppoSAX Meeting, ed van den Heuvel E P J, in 't Zand J J M and Wijers R A M J, in press

[227] Mashhoon B 1985 Gravitational effects of rotating masses *Found. Phys.* **15** 497

[228] Masuda N and Eriguchi Y 1997 Three-dimensional simulations of runaway instability of self-gravitating accretion disks *Astrophys. J.* **489** 804

[229] Masuda N, Nishida S and Eriguchi Y 1998 The runaway instability of self-gravitating tori with non-constant specific angular momentum around black holes *Mon. Not. R. Astron. Soc.* **297** 1139

[230] Matt G 1993 X-ray polarization properties of a centrally illuminated accretion disc *Mon. Not. R. Astron. Soc.* **260** 663

[231] Medvedev M V and Narayan R 2000 Axisymmetric self-similar equilibria of self-gravitating isothermal systems *Astrophys. J.* **541** 579

[232] Meinel R 2000 Gravitational field of rotating disks and black holes *Annalen Phys.* **9** 335

[233] Merloni A and Fabian A C 2003 Iron Kα line profiles and the inner boundary condition of accretion flows *Mon. Not. R. Astron. Soc.* **342** 951

[234] Mestel L 1963 On the galactic law of rotation *Astrophys. J.* **126** 553

[235] Mészáros P 2001 Gamma-ray bursts: accumulating afterglow implications, progenitor clues, and prospects *Science* **291** 79

[236] Mineshige S and Umemura M 1996 Self-similar, self-gravitating viscous disks *Astrophys. J.* **469** L49; 1997 *ibid* **480** 167

[237] Mirabel I F 2003 Jets in supermassive and stellar-mass black holes, in *Proc. XXI Symp. on Relativistic Astrophysics* ed Bandiera R, Maiolino R and Mannucci F (Singapore: World Scientific)

[238] Miralda-Escudé J and Kollmeier J A 2003 Feeding accretion disks in active galactic nuclei with stars crashing on the disk *Astrophys. J.* submitted (astro-ph/0310717)

[239] Miyamoto M and Nagai R 1975 Three-dimensional models for the distribution of mass in galaxies *Publ. Astron. Soc. Japan* **27** 533

[240] Morbidelli A 2002 *Modern Celestial Mechanics: Aspects of Solar System Dynamics* (London: Taylor & Francis)

[241] Morgan T and Morgan L 1969 The gravitational field of a disk *Phys. Rev.* **183** 1097 (Errata: 1969 *ibid* **188** 2544; 1970 *ibid D* **1** 3522)

[242] Müller E and Steinmetz M 1995 Simulating self-gravitating fluids *Computer Phys. Commun.* **89** 45

[243] Nakamura T, Oohara K and Kojima Y 1987 General relativistic collapse to black holes and gravitational waves from black holes *Prog. Theor. Phys. Suppl.* **90** 1

[244] Narayan R 2000 Hydrodynamic drag on a compact star orbiting a supermassive black hole *Astrophys. J.* **536** 663

[245] Narayan R 2002 Why Do AGN Lighthouses Switch Off? in *Lighthouses of the Universe: The Most Luminous Celestial Objects and Their Use for Cosmology* ed Gilfanov M, Sunyaev R

et al (Berlin: Springer) p 405

[246] Narayan R, Paczynski B and Piran T 1992 Gamma-ray bursts as the death throes of massive binary stars *Astrophys. J.* **395** L83

[247] Nayakshin S and Kallman T R 2001 Accretion disk models and their X-ray reflection signatures. I. Local spectra *Astrophys. J.* **546** 406

[248] Nelson R P, Papaloizou J C B, Masset F and Kley W 2000 The migration and growth of protoplanets in protostellar discs *Mon. Not. R. Astron. Soc.* **318** 18

[249] Neugebauer G 1996 Gravitostatics and rotating bodies, in *General Relativity, 46th Scottish Universities Summer School in Physics* ed Hall G S and Pulham J R (London: IOP) p 61

[250] Neugebauer G 2000 Rotating bodies as boundary value problems *Annalen Phys.* **9** 342

[251] Neugebauer G and Meinel R 2003 Progress in relativistic gravitational theory using the inverse scattering method *J. Math. Phys.* **44** 3407

[252] Nishida S and Eriguchi Y 1994 A general relativistic toroid around a black hole *Astrophys. J.* **427** 429

[253] Nishida S and Eriguchi Y 1996 Runaway instability of black hole–neutron torus systems *Astrophys. J.* **461** 320

[254] Nishida S, Lanza A, Eriguchi Y and Abramowicz M A 1996 Runaway instability and gamma-ray bursts *Mon. Not. R. Astron. Soc.* **278** L41

[255] Nolthenius R A and Katz J I 1982 The passage of a star by a massive black hole *Astrophys. J.* **263** 377

[256] Novikov I D, Pethick C J and Polnarev A G 1992 Tidal capture of stars by a massive black hole *Mon. Not. R. Astron. Soc.* **255** 276

[257] Novikov I D and Thorne K S 1973 Astrophysics of black holes, in *Black Holes* ed DeWitt C and DeWitt B S (New York: Gordon and Breach) p 343

[258] Ostriker J 1964 The equilibrium of self-gravitating rings *Astrophys. J.* **140** 1067

[259] Ostriker J P 1983 Viscous drag on an accretion disk due to an embedded stellar system *Astrophys. J.* **273** 99

[260] Paczyński B 1978 A model of selfgravitating accretion disk *Acta Astron.* **28** 91

[261] Paczyński B 1978 A model of selfgravitating accretion disk with a hot corona *Acta Astron.* **28** 241

[262] Paczyński B 1998 Are gamma-ray bursts in star-forming regions? *Astrophys. J.* **494** L45

[263] Paczyński B and Wiita P J 1980 Thick accretion disks and supercritical luminosities *Astron. Astrophys.* **88** 23

[264] Page D N and Thorne K S 1974 Disk-accretion onto a black hole. I. Time-averaged structure of accretion disks *Astrophys. J.* **191** 499

[265] Pandey U S and Gupta A C 1998 Equilibria of self-gravitating gaseous disks *Bull. Astron. Soc. India* **26** 731

[266] Papaloizou J C B and Lin D N C 1995 Theory of accretion disks I: Angular momentum transport processes *Ann. Rev. Astron. Astrophys.* **33** 505

[267] Papaloizou J C B and Pringle J E 1984 The dynamical stability of differentially rotating discs with constant specific angular momentum *Mon. Not. R. Astron. Soc.* **208** 721

[268] Papaloizou J C B and Savonije G J 1991 Instabilities in self-gravitating gaseous discs *Mon. Not. R. Astron. Soc.* **248** 353

[269] Papaloizou J C B, Terquem C and Lin D N C 1998 On the global warping of a thin self-gravitating near-Keplerian gaseous disk with application to the disk in NGC 4258 *Astrophys. J.* **497** 212

[270] Peebles P J E 1972 Star distribution near a collapsed object *Astrophys. J.* **178** 371

[271] Peng H 1983 On calculation of magnetic-type gravitation and experiments *Gen. Rel. Grav.* **15** 725

[272] Perryman M A C 2000 Extra-solar planets *Rep. Prog. Phys.* **63** 1209

[273] Peterson B M 1997 *Introduction to Active Galactic Nuclei* (Cambridge: Cambridge Univ. Press)

[274] Peterson B M 2001 Variability of active galactic nuclei, in *Advanced Lectures on the Starburst-AGN Connection* ed Aretxaga L, Kunth D and Mújica R (Singapore: World Scientific) p 3

[275] Pichon C and Lynden-Bell D 1996 New sources for Kerr and other metrics: rotating relativistic discs with pressure support *Mon. Not. R. Astron. Soc.* **280** 1007

[276] Pineault S and Landry S 1994 Impact of small bodies onto disks around compact objects – I. Impact satellites *Mon. Not. R. Astron. Soc.* **267** 557

[277] Plummer H C 1991 On the problem of distribution in globular star clusters *Mon. Not. R. Astron. Soc.* **71** 460

[278] Polyachenko V L, Polyachenko E V and Strel'nikov A V 1997 Stability criteria for gaseous

self-gravitating disks *Astronomy Lett.* **23** 483

[279] Poutanen J 2001 Understanding spectral variability and time lags in accreting black holes *Advances in Space Research* **28** 267

[280] Pringle J E 1981 Accretion discs in astrophysics *Ann. Rev. Astron. Astrophys.* **19** 137

[281] Pringle J E 1996 Self-induced warping of accretion discs *Mon. Not. R. Astron. Soc.* **281** 357; 1997 *ibid* **292** 136

[282] Pruet J, Woosley S E and Hoffman R D 2002 Nucleosynthesis in gamma ray burst accretion disks *Astrophys. J.* **586** 1254

[283] Punsly B 2001 *Black Hole Gravitohydrodynamics* (Berlin: Springer)

[284] Quevedo H 1992 Generating solutions of the Einstein-Maxwell equations with prescribed physical properties *Phys. Rev. D* **45** 1174

[285] Quevedo H and Mashhoon B 1991 Generalizations of Kerr spacetime *Phys. Rev. D* **43** 3902

[286] Quinlan G D, Hernquist L and Sigurdsson S 1995 Models of galaxies with central black holes: adiabatic growth in spherical galaxies *Astrophys. J.* **440** 554

[287] Quinlan G D and Shapiro S L 1987 The collapse of dense star clusters to supermassive black holes – binaries and gravitational radiation *Astrophys. J.* **321** 199

[288] Quinlan G D and Shapiro S L 1990 The dynamical evolution of dense star clusters in galactic nuclei *Astrophys. J.* **356** 483

[289] Rauch K P 1995 Dynamical evolution of star clusters around a rotating black hole with an accretion disc *Mon. Not. R. Astron. Soc.* **275** 628

[290] Rauch K P 1999 Collisional stellar dynamics around massive black holes in active galactic nuclei *Astrophys. J.* **514** 725

[291] Rauch K P and Tremaine S 1996 Resonant relaxation in stellar systems *New Astronomy* **1** 149

[292] Reynolds C S 1996 *X-Ray emission and absorption in active galaxies*, PhD Thesis (Cambridge Univ., Cambridge)

[293] Reynolds C S and Nowak M A 2003 Fluorescent iron lines as a probe of astrophysical black hole systems *Phys. Rep.* **377** 389

[294] Rice W M, Armitage J, Bate R and Bonnell A 2003 The effect of cooling on the global stability of self-gravitating protoplanetary discs *Mon. Not. R. Astron. Soc.* **339** 1025

[295] Richard D and Zahn J-P 1999 Turbulence in differentially rotating flows. What can be learned from the Couette-Taylor experiment *Astron. Astrophys.* **347** 734

[296] Rosswog S, Davies M B, Thielemann F-K and Piran T 2000 Merging neutron stars: asymmetric systems *Astron. Astrophys.* **360** 171

[297] Różańska A, Dumont A-M, Czerny B and Collin S 2002 The structure and radiation spectra of illuminated accretion discs in active galactic nuclei – I. Moderate illumination *Mon. Not. R. Astron. Soc.* **332** 799

[298] Ruffert M and Janka H-T 1999 Gamma-ray bursts from accreting black holes in neutron star mergers *Astron. Astrophys.* **344** 573

[299] Sakimoto P J and Coroniti F V 1981 Accretion disk models for QSOs and active galactic nuclei – The role of magnetic viscosity *Astrophys. J.* **247** 19

[300] Salow R M and Statler T S 2001 Eccentric nuclear disks with self-gravity: predictions for the double nucleus of M 31 *Astrophys. J.* **551** L49

[301] Sarbach O, Heusler M and Brodbeck O 2001 Self-adjoint wave equations for dynamical perturbations of self-gravitating fields *Phys. Rev. D* **63** 104015

[302] Schinnerer E, Eckart A and Tacconi L J 2001 The nuclear stellar cluster in the Seyfert 1 galaxy NGC 3227: High angular resolution near-infrared imaging and spectroscopy *Astrophys. J.* **549** 254

[303] Schmid-Burgk J 1967 Finite amplitude density variations on a self-gravitating isothermal gas layer *Astrophys. J.* **149** 727

[304] Semerák O 2001 Curvature singularity around first Morgan–Morgan disc *Class. Quantum Grav.* **18** 3589

[305] Semerák O 2002 Expulsion of external fields from extreme horizons: example of an external gravitational field *Czech. J. Phys.* **52** 11

[306] Semerák O 2002 Towards gravitating discs around rotating black holes, in *Gravitation: Following the Prague Inspiration* ed Semerák O, Podolský J and Žofka M (Singapore: World Scientific) p 111

[307] Semerák O 2002 Thin disc around a rotating black hole, but with support in-between *Class. Quantum Grav.* **19** 3829

[308] Semerák O 2003 Static discs around a Schwarzschild black hole: III *Class. Quantum Grav* **20** 1613

[309] Semerák O and Karas V 1999 Pseudo-Newtonian models of a rotating black hole field *Astron.*

Astrophys. **343** 325

[310] Semerák O and Žáček M 2000 Static discs around a Schwarzschild black hole: I *Class. Quantum Grav.* **17** 1613

[311] Semerák O and Žáček M 2000 Oscillations of static disks around Schwarzschild black holes: effect of self-gravitation *Publ. Astron. Soc. Japan* **52** 1067

[312] Semerák O, Zellerin T and Žáček M 1999 The structure of superposed Weyl fields *Mon. Not. R. Astron. Soc.* **308** 691 (Erratum: 2001 *ibid* **322** 207)

[313] Semerák O, Žáček M and Zellerin T 1999 Test-particle motion in superposed Weyl fields *Mon. Not. R. Astron. Soc.* **308** 705

[314] Shakura N I and Sunyaev R A 1973 Black holes in binary systems. Observational appearance *Astron. Astrophys.* **24** 337

[315] Shapiro S L and Marchant A B 1978 Star clusters containing massive, central black holes — Monte Carlo simulations in two-dimensional phase space *Astrophys. J.* **225** 603

[316] Shlosman I and Begelman M C 1987 Self-gravitating accretion disks in active galactic nuclei *Nature* **329** 810

[317] Shlosman I and Begelman M C 1989 Evolution of self-gravitating accretion disks in active galactic nuclei *Astrophys. J.* **341** 685

[318] Shlosman I, Begelman M C and Frank J 1990 The fuelling of active galactic nuclei *Nature* **345** 679

[319] Shore S N and White R L 1982 Self-gravitating accretion disk models for active galactic nuclei: self-consistent α-models for the broad emission-line region *Astrophys. J.* **256** 390

[320] Sibgatullin N R 1991 *Oscillations and Waves in Strong Gravitational and Electromagnetic Fields* (Berlin: Springer)

[321] Sibgatullin N R, Garcia A A and Manko V S 2003 Density distribution in massive galactic disks with a central black hole *Astro. Lett.* **29** 70; 2003 *ibid* **29** 825

[322] Sirko E and Goodman J 2003 Spectral energy distributions of marginally self-gravitating quasi-stellar object disks *Mon. Not. R. Astron. Soc.* **341** 501

[323] Smarr L 1973 Surface geometry of charged rotating black holes *Phys. Rev. D* **7** 289

[324] Smarr L 1973 Mass formula for Kerr black holes *Phys. Rev. Lett.* **30** 71

[325] Stahler S W 1983 The equilibria of rotating isothermal clouds. II. Structure and dynamical stability *Astrophys. J.* **268** 155

[326] Statler T S 2001 A simple family of models for eccentric Keplerian fluid disks *Astron. J.* **122** 2257

[327] Stella L 1990 Measuring black hole mass through variable line profiles from accretion disks *Nature* **344** 747

[328] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions to Einstein's Field Equations, Second Edition* (Cambridge: Cambridge Univ. Press)

[329] Störzer H 1993 Structure and spectra of accretion disks in the innermost parts of active galaxies *Astron. Astrophys.* **271** 25

[330] Šubr L 2001 *Interaction between stars and the environment of galactic nuclei*, PhD Thesis (Charles Univ., Prague)

[331] Šubr L and Karas V 1999 An orbiter crossing an accretion disc *Astron. Astrophys.* **352** 452

[332] Syer D, Clarke C J and Rees M J 1991 star–disc interactions near a massive black hole *Mon. Not. R. Astron. Soc.* **250** 505

[333] Takahashi K and Lee H M 2000 Evolution of multimass globular clusters in the Galactic tidal field with the effects of velocity anisotropy *Mon. Not. R. Astron. Soc.* **316** 671

[334] Tanabe Y 1979 Generation of the Lewis solutions from the Weyl solutions *J. Math. Phys.* **20** 1486

[335] Tassoul J-L 1978 *Theory of Rotating Stars* (Princeton: Princeton Univ. Press)

[336] Tertychniy S I 1990 Some representations of the general solution of the linearised vacuum Einstein equations for axially symmetric stationary metrics *Class. Quantum Grav.* **7** 1345

[337] Teukolsky S A 1973 Perturbations of a rotating black hole. I. Fundamental equations for gravitational, electromagnetic, and neutrino-field perturbations *Astrophys. J.* **185** 635

[338] Teukolsky S A 1998 Black hole collisions, toroidal black holes, and numerical relativity, in *Black Holes and Relativistic Stars* ed Wald R M (Chicago: Univ. Chicago Press) p 123

[339] Thorne K S 1998 Probing black holes and relativistic stars with gravitational waves, in *Black Holes and Relativistic Stars* ed Wald R M (Chicago: Univ. Chicago Press) p 41

[340] Thorne K S, Price R H and Macdonald D A (ed) 1986 *Black Holes: The Membrane Paradigm* (New Haven: Yale Univ. Press)

[341] Toomre A 1964 On the gravitational stability of a disk of stars *Astrophys. J.* **139** 1217

[342] Toomre A 1977 Theories of spiral structure *Ann. Rev. Astron. Astrophys.* **15** 437

[343] Torres del Castillo G F 1990 Generation of perturbations by means of decoupled equations and their adjoints *Gen. Rel. Grav.* **22** 1085

[344] Torres del Castillo G F and Cartas-Fuentevilla R 1996 Scattering by a Reissner-Nordström black hole *Phys. Rev. D* **54** 4886

[345] Torres del Castillo G F and Solís-Rodríguez H G 1999 Self-duality and gravitational perturbations *J. Math. Phys.* **40** 4099

[346] Tsuribe T 1999 Self-similar viscous accretion and growth of a central core in self-gravitating disks *Astrophys. J.* **527** 102

[347] Turakulov Z Ia 1984 Equilibrium of collisionless gravitating sphere and disk *Astrophysics* **19** 448; also 1989 *Int. J. Mod. Phys. A* **4** 3653; 1990 *ibid* **5** 725

[348] Usui F, Nishida S and Eriguchi Y 1998 Emission-line profiles from self-gravitating toroids around black holes *Mon. Not. R. Astron. Soc.* **301** 721

[349] van der Klis M 2000 Millisecond Oscillations in X-ray Binaries *Ann. Rev. Astron. Astrophys.* **38** 717

[350] van Putten M H P M 2001 Gamma-ray bursts: LIGO/VIRGO sources of gravitational radiation *Phys. Rep.* **345** 1

[351] van Putten M H P M and Levinson A 2003 Theory and astrophysical consequences of a magnetized torus around a rapidly rotating black hole *Astrophys. J.* **584** 937

[352] Vein P R 1985 2 related families of determinantal solutions of the stationary axially-symmetric vacuum Einstein equations *Class. Quantum Grav.* **2** 899

[353] Viergutz S U 1993 Image generation in Kerr geometry. I. Analytical investigations on the stationary emitter-observer problem *Astron. Astrophys.* **272** 355

[354] Vilkoviskij E Y and Czerny B 2002 The role of the central stellar cluster in active galactic nuclei. I. Semi-analytical model *Astron. Astrophys.* **387** 804

[355] Vogt D and Letelier P S 2003 Exact general relativistic perfect fluid disks with halos *Phys. Rev. D* **68** 084010

[356] Vokrouhlický D and Karas V 1993 A star orbiting around a supermassive rotating black-hole *Mon. Not. R. Astron. Soc.* **265** 365

[357] Vokrouhlický D and Karas V 1998 Stellar dynamics in a galactic centre surrounded by a massive accretion disc *Mon. Not. R. Astron. Soc.* **298** 53

[358] Wald R M 1984 *General Relativity* (Chicago: Univ. Chicago Press)

[359] Ward W R 1997 Protoplanet migration by nebula tides *Icarus* **126** 261

[360] Will C M 1974 Perturbation of a slowly rotating black hole by a stationary axisymmetric ring of matter. I. Equilibrium configurations *Astrophys. J.* **191** 521

[361] Will C M 1975 Perturbation of a slowly rotating black hole by a stationary axisymmetric ring of matter. II. Penrose processes, circular orbits, and differential mass formulae *Astrophys. J.* **196** 41

[362] Wilms J, Reynolds C S, Begelman M C, Reeves J, Molendi S *et al* 2001 XMM-EPIC observation of MCG-6-30-15: direct evidence for the extraction of energy from a spinning black hole? *Mon. Not. R. Astron. Soc.* **328** L27

[363] Wilson R E 1981 Equilibrium figures for beta Lyrae type disks *Astrophys. J.* **251** 246

[364] Wilson D B 1984 A runaway instability in thick accretion disks? *Nature* **312** 620

[365] Woodward J W, Sankaran S and Tohline J E 1992 Tidal disruption of a star by a massive disk (The axisymmetric Roche problem) *Astrophys. J.* **394** 248

[366] Wozniak H, Combes F, Emsellem E and Friedli D 2003 Numerical simulations of central stellar velocity dispersion drops in disc galaxies *Astron. Astrophys.* **409** 469

[367] Yorke H W, Bodenheimer P and Laughlin G 1993 The formation of protostellar disks. I. 1 $M_\odot$ *Astrophys. J.* **411** 274

[368] Young P 1980 Numerical models of star clusters with a central black hole. I – Adiabatic models *Astrophys. J.* **242** 1232

[369] Young A J, Fabian A C, Ross R R and Tanaka Y 2001 A complete relativistic ionized accretion disk in Cygnus X-1 *Mon. Not. R. Astron. Soc.* **325** 1045

[370] Žáček M and Semerák O 2002 Static discs around a Schwarzschild black hole: II *Czech. J. Phys.* **52** 19

[371] Zanotti O, Rezzolla L and Font J A 2003 Quasi-periodic accretion and gravitational waves from oscillating 'toroidal neutron stars' around a Schwarzschild black hole *Mon. Not. R. Astron. Soc.* **341** 832

[372] Zellerin T and Semerák O 2000 Two-soliton stationary axisymmetric sprouts from Weyl seeds *Class. Quantum Grav.* **17** 5103

[373] Zhao HongSheng, Haehnelt M G and Rees M J 2002 Feeding black holes at galactic centres by capture from isothermal cusps *New Astronomy* **7** 385

[374] Zier C and Biermann P L 2001 Binary black holes and tori in AGN. I. Ejection of stars and merging of the binary *Astron. Astrophys.* **77** 23; 2002 *ibid* **396** 91
[375] Zurek W H, Siemiginowska A and Colgate S A 1994 Star-disk collisions and the origin of the broad lines in quasars *Astrophys. J.* **434** 46; 1996 *ibid* **470** 652